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"On Possibilistic Representations of Fuzzy Intervals"

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On Possibilistic Representations of Fuzzy Intervals

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Abstract

It is well known that a fuzzy interval has two equivalent representations given in terms of the so called Left and Right sides of the membership function (LR-representation) or in terms of the Lower and Upper branches defining the endpoints of the α-cuts (LU-representation).

In this paper we suggest an additional representation of fuzzy intervals called ACF-representation (using an average cumulative function instead of the membership function), based on possibility theory.

We illustrate how to build the new representation and we state its basic properties. The main result is that the Average Cumulative (AC) function can be uniquely defined for any fuzzy interval and it is possible to go from one representation to the others through appropriate transformations.

An interesting link can be established between ACF-representation and quantile functions, with a possible statistical interpretation useful in real application.

We also recommend a parametric form of the AC function.

KEYWORDS: possibility distribution, parametric representations, fuzzy intervals, quantiles, average cumulative function.

1 Introduction

Representation of fuzzy intervals may take advantage of some key concepts emerging from possibility theory. Possibility theory has been widely studied; in particular, for a given normal, upper-semicontinuous and quasi-concave membership function two dual functions, called the possibility and the necessity measures have been introduced by Dubois and Prade in [10] and [11]. The relationship between membership functions and possibility distributions was primary introduced by Zadeh ([38]) in order to provide a graded semantics to natural language statements. Many other aspects have been focused by Dubois and Prade in [12] (see also the recent paper [14]) for normal fuzzy sets; Klir in [22] generalizes the standard fuzzy-set interpretation to non normal fuzzy sets

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too. Furthermore, Dubois in [9] shows that many notions in statistics are well interpreted by the numerical possibility theory and Baudrit and Dubois (details in [1]) analyze the existing relations between possibility theory, imprecise probability and belief functions. More recently, Couso and Sanchez in [5] and [6] rephrase the possibilistic interpretation of fuzzy sets to define fuzzy random variables and confidence interval for fuzzy approximations.

In this paper we present a new representation of fuzzy intervals (called ACF-representation) based on possibility theory: associated to the membership function, we define an Average Cumulative Function (ACF), which is monotonic with values in $[0, 1]$. We illustrate how to build the representation and we analyze its basic properties. The ACF can be uniquely defined for any fuzzy interval and we show that the $\alpha-$cuts of a fuzzy interval u can be directly obtained from the ACF.

A relevant aspect motivating the use of the ACF representation stays in the fact that it can be successfully adopted to determine the membership function from experimental data. An interesting link is indeed established between AC function and quantile functions with a possibly statistical interpretation useful in real applications.

The interest for the ACF-representation is that a biunivocal relationship can be established between the set of ACFs and the membership functions of fuzzy intervals. In the specific case of continuous membership function, the ACF is continuous too (and monotonic) and it has the same properties of a statistical cumulative function. We will shortly discuss about parametric ACF functions, based on monotonic spline functions, in order to obtain families of "basic" functions to work with for computations and applications.

As well known, any fuzzy interval has two equivalent representations given in terms of the so called Left and Right side of the membership function (LR-representation introduced by Dubois Prade) or in terms of Lower and Upper branches defining the endpoints of the α-cuts (LU-representation introduced by Voxman and Goetschel). We will analyze the basic relationship between ACF and LU representations, so obtaining a strict connection between all three LR, LU and ACF representations; it is then possible to go from one representation to the other two, through appropriate transformations. The representations above can be related to the horizontal membership functions introduced in [29] and [28]; we will not consider it here, as in general it is obtained by a convex combination of the $\alpha-$cuts in the LU-representation.

The paper is organized in five sections; in section 2 we define the ACF, its basic properties and its relation with the LU-representation i.e. the $\alpha-$cuts. In section 3 we show the connection between the ACF and the quantile functions and in section 4 the parametric representation of ACF is detailed with examples. The fifth section ends the paper with some ideas for future research.

2 Average Cumulative functions associated to a fuzzy interval

We consider real fuzzy intervals u with compact support $[a,b]$ and compact nonempty core $[c,d] \subset [a,b]$ where $a \le c \le d \le b \in \mathbb{R}$; they are defined in terms of a quasi-concave, upper-semicontinuous function $u : \mathbb{R} \longrightarrow [0,1]$ such that $[a,b] = cl\left(\{x|u(x) > 0\}\right)$ is the support and $[c,d] = \{x|u(x) = 1\}$ is the core (here, $cl\,(A)$ is the closure of set A). The space of fuzzy intervals will be denoted by $\mathbb{R}_{\mathcal{F}}$.

The membership function of $u \in \mathbb{R}_{\mathcal{F}}$ can be represented in the form

$$u(x) = \begin{cases} 0 & \text{if} \quad x < a, \\ u^L(x) & \text{if} \quad a \le x < c, \\ 1 & \text{if} \quad c \le x \le d, \\ u^R(x) & \text{if} \quad d < x \le b, \\ 0 & \text{if} \quad x > b. \end{cases} \tag{1}$$

where $u^L : [a,c] \longrightarrow [0,1[$ is a nondecreasing right-continuous function, $u^L(x) > 0$ for $x \in]a,c]$, called the *left side* of the fuzzy interval and $u^R : [d,b] \longrightarrow [0,1]$ is a nonincreasing left-continuous function, $u^R(x) > 0$ for $x \in [d,b[$, called the *right side* of the fuzzy interval. If $c = d$ then u is called a fuzzy number and $\{c\}$ is the core or u.

We extend the two functions $u^L(x)$ and $u^R(x)$ to the real domain by setting

$$u^L_{ext}(x) = \begin{cases} 0 & if \quad x < a, \\ u^L(x) & if \quad a \le x < c, \\ 1 & if \quad x \ge c, \end{cases}$$

$$u^R_{ext}(x) = \begin{cases} 1 & if \quad x \le d, \\ u^R(x) & if \quad d < x \le b, \\ 0 & if \quad x > b. \end{cases}$$

Following Dubois-Kerre-Mesiar-Prade (see, e.g., [16], [17]), a fuzzy interval $u \in \mathbb{R}_{\mathcal{F}}$ can be viewed as a possibility distribution on the real numbers and there exists a pair of cumulative distribution functions (CDF), called the lower CDF and the upper CDF of u, respectively, based on the extended left side function $u^L_{ext}(x)$ and the extended right side function $u^R_{ext}(x)$. As described in [16], a fuzzy interval u with membership (1) can be equivalently characterized by the pair (Pos_u, Nec_u) of cumulative distribution functions $Pos_u : \mathbb{R} \longrightarrow [0,1]$ and $Nec_u : \mathbb{R} \longrightarrow [0,1]$ given by

$$\begin{aligned} Pos_u(x) &= \sup\{u\,(t) \mid t \le x\} = u^L_{ext}(x) \\ Nec_u(x) &= 1 - \sup\{u\,(t) \mid t > x\} = 1 - \lim_{t \downarrow x} u^R_{ext}(t). \end{aligned}$$

By construction, the two distribution functions Pos_u and Nec_u are nondecreasing and càdlàg (French "continue à droite, limite à gauche", right continuous with left limits) at all points of their domain $\mathbb{R}$.

For interpretations of Pos_u and Nec_u see the extended literature on possibility theory, started with [38], [11], [17]; see also the recent book [2] and the references therein.

In this paper, instead of the pair (Pos_u, Nec_u), we consider a modified pair of functions where the second component is substituted by

$$F_u^R(x) = 1 - u_{ext}^R(x) = \begin{cases} 0 & if \quad x \leq d, \\ 1 - u^R(x) & if \quad d < x \leq b, \\ 1 & if \quad x > b. \end{cases} ;$$

we call F_u^R the quasi-necessity function of u. For uniformity of notation, we also denote

$$F_u^L(x) = u_{ext}^L(x).$$

Remark that, for continuous membership functions, we always have $\lim_{t \downarrow x} u_{ext}^R(t) = u_{ext}^R(x)$ and $F_u^R(x) = Nec_u(x)$.

Any weighted average (convex combination) of the two functions F_u^L and F_u^R can be used to represent a fuzzy interval.

Definition 1 *For a fixed value of $\lambda \in [0,1]$, the λ-Average Cumulative function (λ-ACF) of u is defined to be the following convex combination of F_u^L and F_u^R, for all $x \in \mathbb{R}$,*

$$\begin{aligned} F_u^{(\lambda)}(x) &= (1-\lambda)F_u^L(x) + \lambda F_u^R(x) \\ &= \begin{cases} 0 & if \quad x < a \\ (1-\lambda)u^L(x) & if \quad a \leq x < c \\ 1-\lambda & if \quad c \leq x \leq d \\ 1 - \lambda u^R(x) & if \quad d < x \leq b \\ 1 & if \quad x > b. \end{cases} \end{aligned}$$

For the value $\lambda = \frac{1}{2}$ we denote $F_u^{(\frac{1}{2})}(x)$ simply by $F_u(x)$.

Remark 2 *In the continuous case, i.e. when the membership function of u is continuous, we have*

$$\begin{aligned} F_u^{(\lambda)}(x) &= (1-\lambda)u_{ext}^L(x) + \lambda\left(1 - u_{ext}^R(x)\right) \\ &= (1-\lambda)Pos_u(x) + \lambda Nec_u(x). \end{aligned}$$

As a consequence, for continuous u, each function $F_u^{(\lambda)} = (1-\lambda)F_u^L + \lambda F_u^R$ is nondecreasing and càdlàg for all $\lambda \in [0,1]$; it can be considered as a cumulative distribution function, as indeed $\lim_{x \longrightarrow -\infty} F_u^{(\lambda)}(x) = 0$ and $\lim_{x \longrightarrow +\infty} F_u^{(\lambda)}(x) = 1$; its generalized inverse, defined by $(F_u^{(\lambda)})^{-1}(t) = \inf\{x \in \mathbb{R} | F_u^{(\lambda)}(x) \geq t\} = \sup\{x \in \mathbb{R} | F_u^{(\lambda)}(x) < t\}$ is also called, in the statistical literature, the quantile function of $F_u^{(\lambda)}$.

Remark 3 *The average of the possibility and necessity functions* $\frac{1}{2}Pos_u(x) + \frac{1}{2}Nec_u(x)$ *is called "credibility distribution" by Liu (see [27]); it coincides with* F_u *when* u *is a continuous fuzzy interval.*

In Figure 1 we represent an LR fuzzy interval u and the corresponding functions $F_u^{(\lambda)}$ for some values of $\lambda \in [0,1]$.

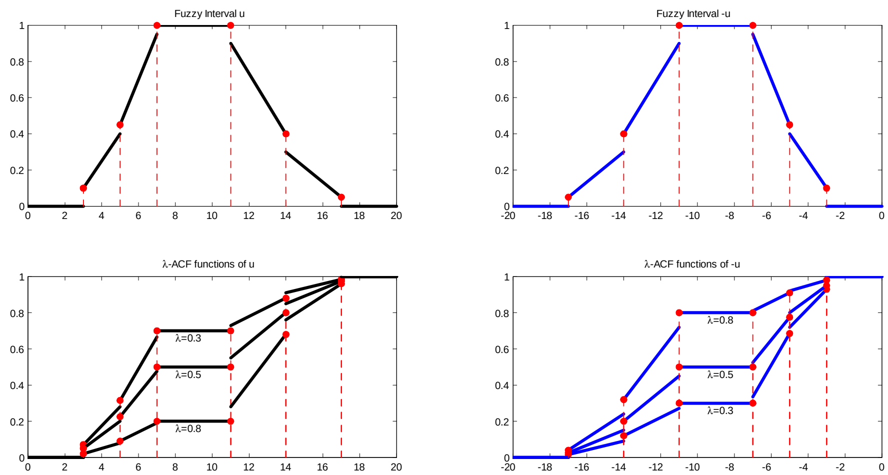

Figure 1. A fuzzy interval u (top,left) and $-u$ (top,right) and the corresponding λ-ACFs, for $\lambda \in \{0.3, 0.5, 0.8\}$.

Proposition 4 *The* λ*-ACF has the following translation property: for a given fuzzy interval* $u \in \mathbb{R}_{\mathcal{F}}$ *and a number* $\rho \in \mathbb{R}$*, the translated fuzzy interval* $v = u + \rho$*, with membership function* $v(x) = u(x - \rho)$*, is such that*

$$\begin{aligned} F_{u+\rho}^{(\lambda)}(x) &= (1-\lambda)v_{ext}^{L}(x) + \lambda v_{ext}^{R}(x) \qquad (2) \\ &= (1-\lambda)u_{ext}^{L}(x-\rho) + \lambda u_{ext}^{R}(x-\rho) \\ &= F_u^{(\lambda)}(x-\rho). \end{aligned}$$

It is interesting to remark a connection between the λ-ACF of u and of the opposite fuzzy interval $-u$; recall that, from the extension principle, the fuzzy number $-u$ can be defined by the following membership function

$$(-u)(x) := u(-x), \text{ for all } x \in \mathbb{R}.$$

We can also write $(-u)(-x) := u(x)$, i.e., the membership value $(-u)(-x)$ of $-x \in \mathbb{R}$, with respect to the fuzzy interval $(-u)$, is the same as the membership value of x with respect to u.

We have the following property, relating the λ-ACF of u and the $(1-\lambda)$-ACF of $-u$.

Lemma 5 *Let $u \in \mathbb{R}_{\mathcal{F}}$ and let $-u \in \mathbb{R}_{\mathcal{F}}$ be its opposite interval; then, the following equality holds true for all $\lambda \in [0,1]$*

$$F_u^{(\lambda)}(-x) + F_{-u}^{(1-\lambda)}(x) = 1, \text{ for all } x \in \mathbb{R}$$

where $F_{-u}^{(1-\lambda)}$ is the $(1-\lambda)$-ACF of $-u$.

Proof. Let $[a,b]$ be the support of u and $[c,d]$ its core, with $a \le c \le d \le b$, so that the support and the core of $-u$ are, respectively, $[-b,-a]$ and $[-d,-c]$, with $-b \le -d \le -c \le -a \le 0$. The membership function of $-u$ is given by

$$(-u)(x) = \begin{cases} 0 & \text{if} \quad x < -b, \\ u^R(-x) & \text{if} \quad -b \le x < -d, \\ 1 & \text{if} \quad -d \le x \le -c, \\ u^L(-x) & \text{if} \quad -c < x \le -a, \\ 0 & \text{if} \quad x > -a \end{cases}$$

so that the extended functions of $-u$ are

$$(-u)^L_{ext}(x) = u^R_{ext}(-x) \tag{3}$$

$$(-u)^R_{ext}(x) = u^L_{ext}(-x) \tag{4}$$

It follows that the two functions $F^L_{-u}(x) = (-u)^L_{ext}(x) = u^R_{ext}(-x)$ $F^R_{-u}(x) = 1-(-u)^R_{ext}(x) = 1-u^L_{ext}(-x)$ are, respectively, nondecreasing, right continuous and nonincreasing, left continuous.
Then, from $F_{-u}^{(1-\lambda)}(x) = \lambda(-u)^L_{ext}(x) + (1-\lambda)\left(1-(-u)^R_{ext}(t)\right)$ with (3)-(4) and $F_u^{(\lambda)}(-x) = (1-\lambda)F_u^L(-x) + \lambda F_u^R(-x)$ we obtain

$$\begin{aligned} F_u^{(\lambda)}(-x) + F_{-u}^{(1-\lambda)}(x) &= (1-\lambda)\,F_u^L(-x) + \lambda F_u^R(-x) \\ &\quad + \lambda F^L_{-u}(-x) + (1-\lambda)\,F^R_{-u}(x) \\ &= (1-\lambda)\,u^L_{ext}(-x) + \lambda\left(1-u^R_{ext}(-x)\right) \\ &\quad + \lambda u^R_{ext}(-x) + (1-\lambda)\left(1-u^L_{ext}(-x)\right) \\ &= \lambda + (1-\lambda) = 1. \end{aligned}$$

■

Remark 6 *From Lemma 5 we immediately deduce the following formula for the λ-ACF of $-u$:*

$$F_{-u}^{(\lambda)}(x) = 1 - F_u^{(1-\lambda)}(-x), \text{ for all } x \in \mathbb{R}.$$

The proved Lemma 5 can now be used to prove our main result, Theorem 7 below: it shows that the α-cuts of any fuzzy interval can be obtained by inverting the λ-ACFs of u and $-u$ for any value of $\lambda \in]0,1[$. After its proof, it will be also immediate to see that the λ-ACFs corresponding to the values $\lambda = 0$ or $\lambda = 1$ are not able to reproduce completely the α-cuts $[u_\alpha^-, u_\alpha^+]$, as in fact $F_u^{(0)}$ and $F_{-u}^{(1)}$ loose information on u^R, while $F_u^{(1)}$ and $F_{-u}^{(0)}$ loose information on u^L.

Before the setting of Theorem 7, we need to point out some facts. As we have seen above, the two functions $F_u^{(\lambda)}$ and $F_{-u}^{(1-\lambda)}$ do not have, in general, the properties of a cumulative distribution function (indeed, $F_u^{(\lambda)}$ is càdlàg on $[a,b]$ only if u^R is continuous and $F_{-u}^{(1-\lambda)}$ is càdlàg on $[-b,-a]$ only if u^L is continuous). But it is immediate to verify that, for any value of $\lambda \in]0,1[$, the function $F_u^{(\lambda)}$ is càdlàg on $[a,d]$ and the function $F_{-u}^{(1-\lambda)}$ is càdlàg on $[-b,-c]$: at least partially in their domains, they have the properties of a cumulative distribution function.

For a given nondecreasing function $F : [a,b] \longrightarrow [0,1]$, the generalized inverse (also called the *quantile function* of F in probability theory, see, e.g. [18], when F is càdlàg) is defined to be the function $F^{-1} :]0,1[\longrightarrow [a,b]$ such that

$$F^{-1}(\alpha) = \inf\{x | F(x) \geq \alpha\} \text{ for all } \alpha \in]0,1[. \tag{5}$$

Remark that an equivalent definition of (5) for a function $F \in \mathbb{F}(\mathbb{R})$, called in [21] the pseudo-inverse $F^{(-)} : [0,1] \longrightarrow \mathbb{R}$ of F, is defined by

$$F^{(-)}(\alpha) = \begin{cases} a_F & \text{if} \quad \{x|F(x) < \alpha\} \text{ is empty} \\ \sup\{x|F(x) < \alpha\} & \text{otherwise.} \end{cases} \tag{6}$$

Several properties of the pseudo inverse of a nondecreasing function are analyzed in [21]. The equivalence between the two definitions in (5) and (6) can be deduced from Theorem 1 in [19], observing that in its proof F is only required to be nondecreasing.

Clearly, F^{-1} is not the ordinary inverse, unless F itself in strictly increasing from 0 to 1. The following well-known properties of F^{-1} (see [18]) have a role in a possibly statistical interpretation of the next Theorem 7:

1) F^{-1} is nondecreasing, left continuous and has right limits $\lim\limits_{h\downarrow 0} F^{-1}(p+h) = \inf\{x|F(x) > p\}$;

2) $F^{-1}(F(x)) \leq x$ for all $x \in [a,b]$ with $0 < F(x) < 1$;

3) $F(F^{-1}(p)) \geq p$ for all $p \in]0,1[$ and for real x it is $F^{-1}(p) \leq x$ if and only if $p \leq F(x)$;

4) (see [18]) if ω is a random variable with uniform distribution on $[0,1]$ and we consider the quantile transformation $\xi = F^{-1}(\omega)$, then the random variable ξ has exactly cumulative distribution function F.

The main theorem below shows that the (partial) càdlàg property 1) is anyhow sufficient to determine all the relevant α-cuts $[u_\alpha^-, u_\alpha^+]$ of u, i.e., for $\alpha \in]0,1]$. Recall that the fuzzy interval $-u$ has α-cuts given by $[-u_\alpha^+, -u_\alpha^-]$, so that, in particular, $u_\alpha^+ = -(-u)_\alpha^-$.

Theorem 7 *Let $u \in \mathbb{R}_{\mathcal{F}}$ and let $F_u^{(\lambda)}$, $F_{-u}^{(1-\lambda)}$ be the λ-ACF of u and the $(1-\lambda)$-ACF of $-u$, respectively, for any given value $\lambda \in]0,1[$. For all $\alpha \in]0,1]$, the α-cut $[u_\alpha^-, u_\alpha^+]$ of u is given by*

$$\begin{aligned} u_\alpha^- &= \inf\left\{x \in [a,c] | F_u^{(\lambda)}(x) \geq (1-\lambda)\alpha\right\} \qquad (7) \\ &= \left(F_u^{(\lambda)}|_*\right)^{-1}((1-\lambda)\alpha) \\ u_\alpha^+ &= -(-u)_\alpha^- = -\inf\left\{x \in [-b,-d] | F_{-u}^{(1-\lambda)}(x) \geq \lambda\alpha\right\} \qquad (8) \\ &= -\left(F_{-u}^{(1-\lambda)}|_*\right)^{-1}(\lambda\alpha) \end{aligned}$$

where $\left(F_u^{(\lambda)}|_*\right)^{-1}$ and $\left(F_{-u}^{(1-\lambda)}|_*\right)^{-1}$ are the generalized inverses of the restrictions of $F_u^{(\lambda)}$ and $F_{-u}^{(1-\lambda)}$ to the subintervals $[a,c]$ and $[-b,-d]$, respectively (or more generally to $]-\infty, c]$ and $]-\infty,-d]$).
In the particular case of $\lambda = \frac{1}{2}$, we obtain, denoting $F_u = F_u^{(\frac{1}{2})}$ and $F_{-u} = F_{-u}^{(\frac{1}{2})}$,

$$\begin{aligned} u_\alpha^- &= \inf\left\{x | F_u(x) \geq \frac{\alpha}{2}\right\} = (F_u)^{-1}\left(\frac{\alpha}{2}\right) \\ u_\alpha^+ &= -\inf\left\{x | F_{-u}(x) \geq \frac{\alpha}{2}\right\} = -(F_{-u})^{-1}\left(\frac{\alpha}{2}\right). \end{aligned}$$

Proof. Let $\alpha \in]0,1]$ be fixed. Observe first that $F_u^{(\lambda)}(x) = 1-\lambda$ for all $x \in [c,d]$ so that $\inf\{x|F_u^{(\lambda)}(x) \geq (1-\lambda)\alpha\} \leq c$ and $-\inf\{x|F_{-u}^{(1-\lambda)}(x) \geq 1-\lambda\alpha\} \leq -d$. With reference to (7) we can consider only $x \leq c$; then inequality $F_u^{(\lambda)}(x) \geq (1-\lambda)\alpha$ means $(1-\lambda)u^L(x) \geq (1-\lambda)\alpha$, i.e., $u^L(x) \geq \alpha$ if $x \in]a,c]$; it follows that $\inf\{x|F_u^{(\lambda)}(x) \geq (1-\lambda)\alpha\} = \inf\{x|u^L(x) \geq \alpha\} = u_\alpha^-$. Analogously, with reference to (8), we can consider only $x \leq -d$, so that inequality $F_{-u}^{(1-\lambda)}(x) \geq \lambda\alpha$, using Lemma 5, means $\lambda u^R(-x) \geq \lambda\alpha$, i.e., $u^R(-x) \geq \alpha$; it follows that $-\inf\left\{x|F_{-u}^{(1-\lambda)}(x) \geq \lambda\alpha\right\} = \sup\{-x|F_{-u}^{(1-\lambda)}(x) \geq \lambda\alpha\} = \sup\{y|u^R(y) \geq \alpha\} = u_\alpha^+$. ■

Remark 8 *A consequence is that, if $F_u(x)$ is continuous and strictly increasing, u_α^- is such that $F_u(u_\alpha^-) = \frac{\alpha}{2}$ and u_α^+ is such that $F_u(u_\alpha^+) = 1-\frac{\alpha}{2}$; furthermore, if u has $\{c\}$ as the core and considering $\alpha = 1$, we obtain $c = \inf\{x|F_u(x) \geq \frac{1}{2}\} = \sup\{x|F_u(x) \leq \frac{1}{2}\}$ i.e. $c = \{x|F_u(x) = \frac{1}{2}\}$. The core value c (assumed to be unique) has the same property as the median of $F_u(x)$, when we consider F_u itself as a probability (statistical) distribution function.*

Consider any nondecreasing function $F : \mathbb{R} \longrightarrow [0,1]$ satisfying the properties below, given any fixed value $\lambda \in]0,1[$:

1) $a_F = \sup\{x|F(x) = 0\} \in \mathbb{R}$, $b_F = \inf\{x|F(x) = 1\} \in \mathbb{R}$ (clearly $a_F \leq b_F$);

2) $c_F = \inf\{x|F(x) \geq 1-\lambda\} \in \mathbb{R}$, $d_F = \sup\{x|F(x) \leq 1-\lambda\} \in \mathbb{R}$ (clearly $c_F \leq d_F$);

3) $a_F \leq c_F \leq d_F \leq b_F$ and F is right-continuous on $[a_F, c_F[$, left-continuous on $]d_F, b_F]$ and $F(x) = 1 - \lambda$ for all $x \in [c_F, d_F]$.

Then there exists a unique fuzzy interval $u_F \in \mathbb{R}_{\mathcal{F}}$ with λ-ACF, for $\lambda \in]0,1[$ given by F. Indeed, the membership function of u_F is given by (compare with Definition 1.)

$$u_F(x) = \begin{cases} 0 & \text{if} \quad x < a_F, \\ \frac{1}{1-\lambda}F(x) & \text{if} \quad a_F \leq x < c_F, \\ 1 & \text{if} \quad c_F \leq x \leq d_F, \\ \frac{1}{\lambda}(1 - F(x)) & \text{if} \quad d_F < x \leq b_F, \\ 0 & \text{if} \quad x > b_F \end{cases}, \tag{9}$$

and, from the assumptions 1), 2) and 3) on F, u_F is a fuzzy interval (the proof is immediate by directly verifying that $u_F \in \mathbb{R}_{\mathcal{F}}$).

We will denote by $\mathbb{F}_\lambda(\mathbb{R})$ the family of all functions $F : \mathbb{R} \longrightarrow [0,1]$ with the properties 1)-2)-3).

We have immediately that, for any fixed $\lambda \in]0,1[$ there exists a bijection between the set of fuzzy intervals, $\mathbb{R}_{\mathcal{F}}$, and the family of nondecreasing functions $\mathbb{F}_\lambda(\mathbb{R})$:

$$\phi_\lambda \quad : \quad \mathbb{R}_{\mathcal{F}} \longrightarrow \mathbb{F}(\mathbb{R}),\ \phi_\lambda(u) = F_u^{(\lambda)} \tag{10}$$

$$\phi_\lambda^{-1} \quad : \quad \mathbb{F}_\lambda(\mathbb{R}) \longrightarrow \mathbb{R}_{\mathcal{F}},\ \phi_\lambda^{-1}(F) = u_F. \tag{11}$$

By construction, it is obvious that $\phi_\lambda^{-1}(\phi_\lambda(u)) = u$ for all $u \in \mathbb{R}_{\mathcal{F}}$, and that $\phi_\lambda(\phi_\lambda^{-1}(F)) = F$ for all $F \in \mathbb{F}_\lambda(\mathbb{R})$. We summarize the above result by the following proposition:

Proposition 9 *For any $u \in \mathbb{R}_{\mathcal{F}}$, its λ-AC function given by definition 1 satisfies properties 1),2),3) and, viceversa, for any F satisfying 1),2),3) there exists a unique element $u_F \in \mathbb{R}_{\mathcal{F}}$, given by (9), having F as its λ-AC function.*

In the particular case of $\lambda = \frac{1}{2}$, the family $\mathbb{F}_\lambda(\mathbb{R})$ will be simply denoted by $\mathbb{F}(\mathbb{R})$ and the bijection ϕ_λ will be denoted by ϕ; the $\frac{1}{2}-$AC function of $u \in \mathbb{R}_{\mathcal{F}}$ is a nondecreasing function $F_u : \mathbb{R} \longrightarrow [0,1]$ such that $F_u(x) = \frac{1}{2}u^L(x)$ on $[a, c[$, $F_u(x) = 1 - \frac{1}{2}u^R(x)$ on $]d, b]$ and $F_u(x) = \frac{1}{2}$ on the core $[c, d]$ of u. Viceversa, if $F \in \mathbb{F}(\mathbb{R})$ is given, the membership function of the corresponding fuzzy number $u_F \in \mathbb{R}_{\mathcal{F}}$ has left and right branches given by $u^L(x) = 2F(x)$ and $u^R(x) = 2 - 2F(x)$

If $u \in \mathbb{R}_{\mathcal{F}}$ is continuous, then $F_u \in \mathbb{F}(\mathbb{R})$ is also continuous; viceversa, if $F \in \mathbb{F}(\mathbb{R})$ is continuous, then also $u_F \in \mathbb{R}_{\mathcal{F}}$ is continuous. So, the bijection $\phi : \mathbb{R}_{\mathcal{F}} \longrightarrow \mathbb{F}(\mathbb{R})$ transforms continuous fuzzy intervals into continuous $\frac{1}{2}$-AC functions and the bijection $\phi^{-1} : \mathbb{F}(\mathbb{R}) \longrightarrow \mathbb{R}_{\mathcal{F}}$ transforms continuous $F \in \mathbb{F}(\mathbb{R})$ into continuous $u_F \in \mathbb{R}_{\mathcal{F}}$.

2.1 Arithmetic operations with ACF

It is interesting to express fuzzy arithmetic operations in terms of $\frac{1}{2}$-AC function. If $\circledcirc$ is any binary operation defined on the space $\mathbb{R}_{\mathcal{F}}$, there exists a corresponding operation $\circledcirc'$ on the space $\mathbb{F}(\mathbb{R})$, such that, in terms of bijections ϕ (10) and ϕ^{-1} (11) it is

$$F_u \circledcirc' F_v = \phi(u \circledcirc v), \text{ and } u \circledcirc v = \phi^{-1}(F_u \circledcirc' F_v). \tag{12}$$

We show how the relations in (12) work for linear shaped trapezoidal fuzzy numbers such as $u = \langle a, c, d, b \rangle$ and $v = \langle a', c', d', b' \rangle$. We know that

$$\lambda u = \begin{cases} \langle \lambda a, \lambda c, \lambda d, \lambda b \rangle & \text{if} \quad \lambda \geq 0 \\ \\ \langle \lambda b, \lambda d, \lambda c, \lambda a \rangle & \text{if} \quad \lambda < 0 \end{cases} \tag{13}$$

and

$$u + v = \langle a + a', c + c', d + d', b + b' \rangle . \tag{14}$$

Corresponding to the left and right branches of the membership functions, the ACF of u is $F_u(x) = \frac{1}{2}\frac{x-a}{c-a}$ on $[a, c[$, $F_u(x) = 1 - \frac{1}{2}\frac{b-x}{b-d}$ on $]d, b]$ (and similarly for v).

Then

$$F_u^{-1}(\alpha) = \begin{cases} a + 2\alpha\,(c-a) & \text{if} \quad 0 \leq \alpha \leq \frac{1}{2} \\ d - 2\left(\frac{1}{2} - \alpha\right)(b-d) & \text{if} \quad \frac{1}{2} < \alpha \leq 1 \end{cases} \tag{15}$$

(similarly for $F_v^{-1}(\alpha)$).

For the scalar multiplication, when $\alpha \in [0, 1]$, it holds

$$F_{\lambda u}^{-1}(\alpha) = \lambda F_u^{-1}(\alpha)$$

and we conclude, according to (13), that

$$F_{\lambda u}^{-1}(\alpha) = \lambda F_u^{-1}(\alpha) = \begin{cases} \lambda a + 2\alpha\lambda\,(c-a) & \text{if} \quad 0 \leq \alpha \leq \frac{1}{2} \\ \lambda d - 2\lambda\left(\frac{1}{2} - \alpha\right)(b-d) & \text{if} \quad \frac{1}{2} < \alpha \leq 1 \end{cases}, \tag{16}$$

if $\lambda \geq 0$, or

$$F_{\lambda u}^{-1}(\alpha) = \lambda F_u^{-1}(\alpha) = \begin{cases} \lambda b - 2\lambda\left(\frac{1}{2} - \alpha\right)(d-b) & \text{if} \quad 0 \leq \alpha \leq \frac{1}{2} \\ \lambda a + 2\alpha\lambda(c-a) & \text{if} \quad \frac{1}{2} < \alpha \leq 1 \end{cases}, \tag{17}$$

if $\lambda < 0$.

For the addition we have, according to (14):

$$\begin{aligned} F_{u+v}^{-1}(\alpha) &= F_u^{-1}(\alpha) + F_v^{-1}(\alpha) = \\ &= \begin{cases} a + a' + 2\alpha\,(c + c' - a - a') & \text{if} \quad 0 \leq \alpha \leq \frac{1}{2} \\ d + d' - 2\left(\frac{1}{2} - \alpha\right)(b + b' - d - d') & \text{if} \quad \frac{1}{2} < \alpha \leq 1 \end{cases}. \end{aligned}$$

3 Average Cumulative function as a quantile function

In the rest of the paper, we consider the λ-ACF functions only for $\lambda = \frac{1}{2}$ and we denote the corresponding $\frac{1}{2}$-ACF of fuzzy intervals u and $-u$ by $F_u(x)$, $F_{-u}(x)$, respectively.

For a real random variable ξ with cumulative distribution function F_ξ, a *quantile* of order $p \in]0,1[$ (or corresponding to the probability p) is a real value x where F_ξ crosses or jumps over p.

Definition 10 *A quantile of order $p \in]0,1[$ for a cumulative distribution function F_ξ (or for the associated random variable ξ) is a real value κ_p such that*

$$\lim_{x \uparrow \kappa_p} F_\xi(x) \le p \quad and \quad F_\xi(\kappa_p) \ge p.$$

According to Theorem 7 and following the extended literature on statistical quantiles it is immediate to deduce the following proposition.

Proposition 11 *Let $u \in \mathbb{R}_\mathcal{F}$ with $\frac{1}{2}$-ACF $F_u(x)$, $x \in \mathbb{R}$; then for all $\alpha \in]0,1]$, the α-cuts $[u_\alpha^-, u_\alpha^+]$ of u are such that u_α^- is the $\frac{\alpha}{2}$-quantile of $F_u(x)$ and $-u_\alpha^+$ is the $\frac{\alpha}{2}$-quantile of $F_{-u}(x)$.*

The proposition above is useful to estimate the level cuts (and the membership function) of a normal fuzzy number or interval $u \in \mathbb{R}_\mathcal{F}$.

Remark that $F_u(c) = \frac{1}{2}$. If $F_u(x)$ is continuous and strictly increasing, u_α^- is such that $F_u(u_\alpha^-) = \frac{\alpha}{2}$ and u_α^+ is such that $F_u(u_\alpha^+) = 1 - \frac{\alpha}{2}$; furthermore, considering $\alpha = 1$, we obtain $c = \inf\{x | F_u(x) \ge \frac{1}{2}\} = \sup\{x | F_u(x) \le \frac{1}{2}\}$ i.e. $c = \{x | F_u(x) = \frac{1}{2}\}$.

Let us consider the case when the membership function is given at a finite number of points, i.e. suppose that the fuzzy number $u \in \mathbb{R}_\mathcal{F}$ is "measured" at N observations $(x_i, u(x_i))$.

We can assume that $a = \min\{x_i | u(x_i) > 0\}$ and $b = \max\{x_i | u(x_i) > 0\}$, corresponding to the values x_i with $u(x_i) > 0$.

For a value of $\alpha \in]0,1]$, the (empirical) $\frac{\alpha}{2}$-quantile and $(1 - \frac{\alpha}{2})$-quantile are obtained by minimizing (see [24], [25] and [37]) the two (empirical) functionals

$$\begin{aligned} S_\alpha^-(m) &= (1 - \frac{\alpha}{2}) \sum_{\substack{i=1 \\ x_i < m}}^{N} (m - x_i) + \frac{\alpha}{2} \sum_{\substack{i=1 \\ x_i > m}}^{N} (x_i - m) \\ S_\alpha^+(m) &= \frac{\alpha}{2} \sum_{\substack{i=1 \\ x_i < m}}^{N} (m - x_i) + (1 - \frac{\alpha}{2}) \sum_{\substack{i=1 \\ x_i > m}}^{N} (x_i - m). \end{aligned}$$

The obtained values

$$m_\alpha^-(N) = \arg\min_m S_\alpha^-(m) \tag{18}$$

$$m_\alpha^+(N) = \arg\min_m S_\alpha^+(m) \tag{19}$$

are an estimate $[m_\alpha^-(N), m_\alpha^+(N)]$ of the α-cut $[u_\alpha^-, u_\alpha^+]$ of u and are obtained without computing directly the (empirical) AC function from the data.

The Glivenko Cantelli theorem can be applied to analyze the convergence of $m_\alpha^-(N)$ and $m_\alpha^+(N)$ to the $\alpha-$cut $[u_\alpha^-, u_\alpha^+]$.

Consider a decomposition $\mathbb{P}_N=\{x_1 \le x_2 ... \le x_N\}$ of the support $[a_F, b_F]$ and define the corresponding empirical AC function as:

$$\widehat{F}_{\mathbb{P}_N}(x) = \frac{1}{N}\sum_{i=1}^{N} I(x \ge x_i)$$

where

$$I(x \ge x_i) = \left\{ \begin{array}{ccc} 1 & if & x \ge x_i \\ \\ 0 & if & x < x_i \end{array} \right\}.$$

Theorem 12 *(Glivenko-Cantelli, chapter 2 in [36]) For a given $\widehat{F}_{\mathbb{P}_N}$ the following holds:*

$$\sup_{x\in[a_F,b_F]} \left|\widehat{F}_{\mathbb{P}_N}(x) - F(x)\right| \xrightarrow{a.s.} 0$$

where the limit is obtained as the mesh length $|\mathbb{P}_N|$ tends to zero.

For a given empirical AC function $\widehat{F}_{\mathbb{P}_N}$ the values $m_\alpha^-(N)$ and $m_\alpha^+(N)$ are called the plug-in non parametric estimators of u_α^- and u_α^+ respectively (as in [36]).

The Glivenko-Cantelli theorem ensures that $\widehat{F}_{\mathbb{P}_N}$ converges to F almost surely and this suggests that the intervals $[m_\alpha^-(N), m_\alpha^+(N)]$ will converge to $[u_\alpha^-, u_\alpha^+]$ as $N \longrightarrow \infty$.

Remark 13 *In several statistical software procedures, the quantiles of the empirical distribution of observations x_i, $i = 1, ..., N$, are frequently computed by sorting the data in ascending order, taking the sorted values $x_{(i)}$ as the quantiles corresponding to probability $p_i = \frac{2i-1}{2N}$, $i = 1, , ..., N$ and using linear interpolation for quantiles corresponding to intermediate probabilities. A similar algorithm is not exact; this is why we adomp (18) and (19). An example is the following with $N = 8$, $x_1 = 2$, $x_2 = 4$, $x_3 = 5$, $x_4 = 6$, $x_5 = 8$, $x_6 = 9$, $x_7 = 11$, $x_8 = 13$; the empirical fuzzy interval has (piecewise constant) membership function given by*

$$u(x) = \begin{cases} 0 & if \quad x < 2 \text{ or } x > 13 \\ \frac{1}{4} & if \quad x \in [2, 4[\text{ or } x \in]11, 13] \\ \frac{1}{2} & if \quad x \in [4, 5[\text{ or } x \in]9, 11] \\ \frac{3}{4} & if \quad x \in [5, 6[\text{ or } x \in]8, 9] \\ 1 & if \quad x \in [6, 8] \text{ (the core)} \end{cases}.$$

In order to evaluate the applicability of equations (18) and (19) to approximate the alfa cuts of a fuzzy number we show a series of five experiments.

We consider the fuzzy number $u \in \mathbb{R}_{\mathcal{F}}$ having α-cuts

$$[u_\alpha^-, u_\alpha^+] = [10\alpha^{0.5}, 12 - 2\alpha^{1.5}],\ \alpha \in [0,1].$$

The core of u is $c = 10$ and the support is $[0, 12]$. Observations from u can be generated by sampling u_α^- and u_α^+ at n values α_i, $i = 1, ..., n$, uniformly between 0 and 1, for different values of n (so than a total of $N = 2n$ data are obtained). Furthermore, to verify robustness, we apply (18),(19) to randomly generated fuzzy numbers $u^{(k)}$, $k = 1, ..., K$ by perturbing the core and/or the support of u, for a given number K of replications.

In each experiment, for a fixed $n \in \{11, 21, 51, 101\}$, $K = 100$ random samples $[(u^{(k)})_i^-, (u^{(k)})_i^+]$ are generated ($i = 1, ..., n$, $k = 1, ..., K$) and (18-19) are applied for each k to obtain $L = 41$ estimated *level*-cuts $[\widehat{u}_{\beta_j}^{(k)-}, \widehat{u}_{\beta_j}^{(k)+}]$ of u, with $\beta_j = \frac{j-1}{L-1}$ ($j = 1, ..., L$); finally, the averages $(\overline{u})_{\beta_j}^- = \frac{1}{K}\sum_{k=1}^K \widehat{u}_{\beta_j}^{(k)-}$ and $(\overline{u})_{\beta_j}^+ = \frac{1}{K}\sum_{k=1}^K \widehat{u}_{\beta_j}^{(k)+}$ are compared with the exact α-cuts of u, $u_{\beta_j}^- = 10\beta_j^{0.5}$, $u_{\beta_j}^+ = 12 - 2\beta_j^{1.5}$. The percentage average absolute error (being $u_{\beta_j}^- \geq 0$, the denominator is set to $1 + u_{\beta_j}^-$ to avoid possible division by zero)

$$AERR = \frac{100}{2L}\sum_{j=1}^{L}\left(\left|\frac{(\overline{u})_{\beta_j}^- - u_{\beta_j}^-}{1 + u_{\beta_j}^-}\right| + \left|\frac{(\overline{u})_{\beta_j}^+ - u_{\beta_j}^+}{1 + u_{\beta_j}^+}\right|\right).$$

In all figures that report the results, the top subplot shows the $K = 100$ replications of the sampled observation points $\left((u^{(k)})_i^-, \alpha_i^{(k)}\right)$ and $\left((u^{(k)})_i^+, \alpha_i^{(k)}\right)$, for $i = 1, ..., n$ and for the selected n (i.e. the membership functions of each $u^{(k)}$). The bottom-left subplot reproduces the analogous points $\left((\widehat{u}^{(k)})_j^-, \beta_j^{(k)}\right)$ and $\left((\widehat{u}^{(k)})_j^+, \beta_j^{(k)}\right)$, i.e. the K membership functions of the fuzzy numbers obtained by (18-19). Finally, the bottom-right subplot reproduces the estimated membership functions (black crosses) and the original membership corresponding to the β_j-cuts, $j = 1, ..., L$ (blue circles).

In the first experiment, the samples $[(u^{(k)})_i^-, (u^{(k)})_i^+]$ are obtained from u simply for different selections of the "observed" levels $\alpha_i^{(k)}$, $i = 1, ..., n$, randomly generated from a uniform distribution between 0 and 1, and replicated K times.

In figure 2, the case with $n = 21$ is pictured.

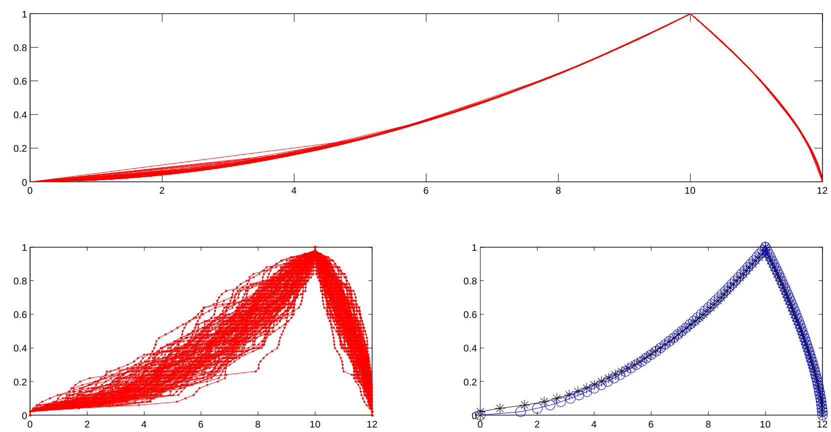

Fig. 2. First experiment, $n = 21$.

In table 1, the AERR is given for the different values of n; we see that it decreases quickly as the number of the observed levels increases from $n = 11$ to $n = 101$.

Table 1: AERR for different values on n in first experiment.

n	11	21	51	101
$AERR$	4.33%	2.26%	0.90%	0.36%

In the next four experiments, the original fuzzy number u is perturbed such that

1. the core is not the constant $c = 10$, but $c^{(k)} = c + \xi_{c,k}$ where each $\xi_{c,k}$ is a normal variable with distribution $N(0, \sigma_c)$ (mean 0 and variance σ_c^2);

2. the left value of the support is not the constant $a = 0$, but $a^{(k)} = \xi_{a,k}$, where each $\xi_{a,k}$ is a normal variable randomly generate from a distribution $N(0, \sigma_a)$ (mean 0 and variance σ_a^2);

3. the right value of the support is not the constant $b = 12$, but $b^{(k)} = 12 + \xi_{b,k}$, where each $\xi_{b,k}$ is a normal variable with distribution $N(0, \sigma_b)$ (mean 0 and variance σ_b^2).

As for the first experiment, we sample u_α^- and u_α^+ at n points α_i generated from uniform distribution on $[0, 1]$ as follows: $\alpha_1 = 0$, $\alpha_i = rand()$ and $\alpha_n = 1$ ($N = 2n$ and $rand()$ is a uniform pseudo-random number generator); the data are then computed with the same shape as for u but with the modified core $c^{(k)}$ and support $[a^{(k)}, b^{(k)}]$ (provided that $a^{(k)} < c^{(k)} < b^{(k)}$)

$$
\begin{aligned}
(u^{(k)})_i^- &= a^{(k)} + (c^{(k)} - a^{(k)})\alpha_i^{0.5} \\
(u^{(k)})_i^+ &= b^{(k)} - (b^{(k)} - a^{(k)})\alpha_i^{1.5}
\end{aligned}
$$

The second experiment uses $\sigma_c = 1.0$, $\sigma_a = 0.0$ and $\sigma_b = 0.0$, i.e. only the core is perturbed. Consider that $\sigma_c = 1.0$ produces a relatively big perturbation with respect to $c = 10.0$. This appears in figure 3, where the case with $n = 21$ is pictured.

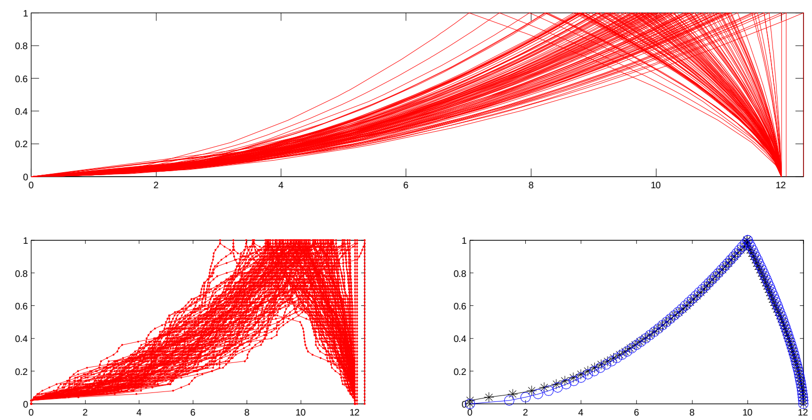

Fig. 3. Second experiment, with $\sigma_c = 1.0$, $\sigma_a = 0.0$, $\sigma_b = 0.0$ and $n = 21$.

In table 2, the AERR for the second experiment is given for the different values of n; also for this experiment the AERR rapidly decreases for $n = 11$ to $n = 101$.

Table 2: AERR for different values on n in second experiment.

n	11	21	51	101
$AERR$	4.38%	2.29%	0.90%	0.35%

The third experiment uses $\sigma_c = 1.0$, $\sigma_a = 0.5$ and $\sigma_b = 1.0$, i.e. the core and the support are both changed with relatively big perturbations. This appears in figure 4, where the case with $n = 21$ is pictured, and in table 3, for different values of n.

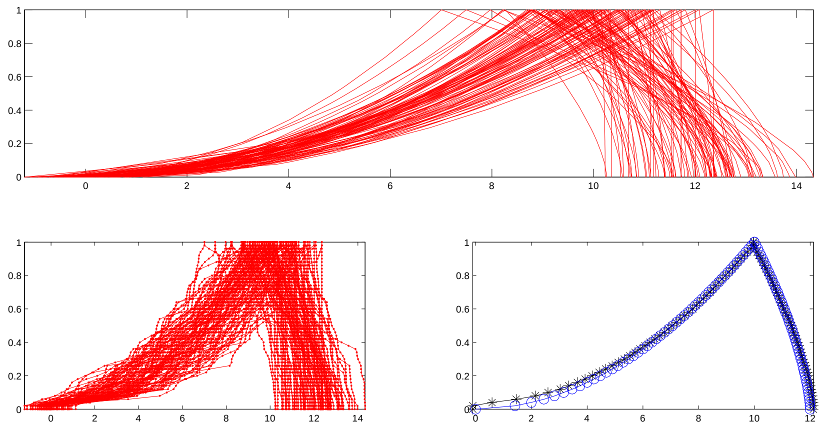

Fig. 4. Third experiment, with $\sigma_c = 1.0$, $\sigma_a = 0.5$, $\sigma_b = 1.0$ and $n = 21$.

Table 3: AERR for different values on n in third experiment.

n	11	21	51	101
$AERR$	4.81%	2.71%	1.26%	0.81%

Finally, the last two experiments are obtained by progressively reducing the perturbations of the core and the support; in the fourth case, we chose $\sigma_c = 0.5$, $\sigma_a = 0.25$ and $\sigma_b = 0.5$, in the fifth case we chose $\sigma_c = 0.2$, $\sigma_a = 0.1$ and $\sigma_b = 0.2$. Tables 4 and 5 give the corresponding $AERR$ for different n.

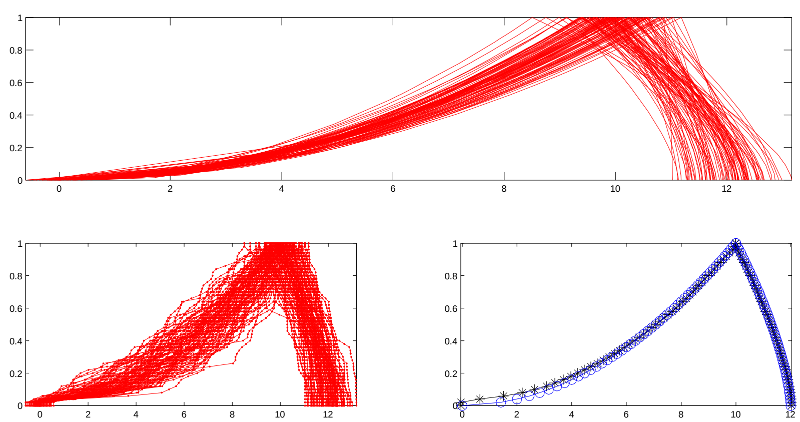

Fig. 5. Fourth experiment, with $\sigma_c = 0.5$, $\sigma_a = 0.25$, $\sigma_b = 0.5$ and $n = 21$.

Table 4: AERR for different values on n in fourth experiment.

n	11	21	51	101
$AERR$	4.54%	2.42%	1.03%	0.46%

.

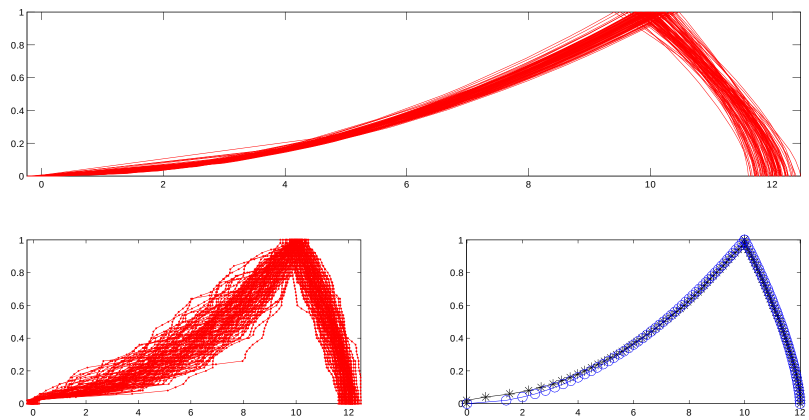

Fig. 6. Fifth experiment, with $\sigma_c = 0.2$, $\sigma_a = 0.1$, $\sigma_b = 0.2$ and $n = 21$.

Table 5: AERR for different values on n in fifth experiment.

n	11	21	51	101
$AERR$	4.41%	2.32%	0.94%	0.37%

.

Resuming the results of the five experiments it follows that the AERR has the same order of magnitude independently from the perturbations we apply and this may be viewed as a good robustness property.

4 Parametric representation of ACFs

As in section 2, we consider fuzzy intervals $u \in \mathbb{R}_{\mathcal{F}}$ with membership function $u : \mathbb{R} \longrightarrow [0,1]$, with compact support $[a,b]$ (where $a = \inf\{x|u(x) > 0\}$ and $b = \sup\{x|u(x) > 0\}$) and nonempty compact core $[c,d]$, $c \leq d$ (where $c = \inf\{x|u(x) = 1\}$ and $d = \sup\{x|u(x) = 1\}$).

From the same section it comes to light that any $u \in \mathbb{R}_{\mathcal{F}}$ has tree parametric representations:

► LR-parametric u_{LR}, with decompositions of the support for left and right sides;

▶ LU-parametric u_{LU}, with decompositions of $[0,1]$ for the lower and upper branches;

▶ ACF-parametric u_{ACF}, i.e.,

$$u_{ACF} = \{(x_i^L, F_i^L), (x_j^R, F_j^R) | i = 0, 1, ..., N_L, j = 0, 1, ..., N_R\}$$

with decompositions of left and right subintervals for *quasi-càdlàg* function F_u. Using appropriate transformations of the parameters we can obtain each one from the other; for example from LR we can obtain ACF as in (**??**).

The distinctions between the various definitions of "inverse" functions for $F \in \mathbb{F}(\mathbb{R})$ as discussed at the end of section 2 again, are no more required if we assume that F is continuous, so that also $u_F \in \mathbb{R}_{\mathcal{F}}$ is continuous.

In the rest of this paper, we assume that F is continuous and we denote by $\mathbb{F}_c(\mathbb{R})$ the subfamily of continuous functions of $\mathbb{F}(\mathbb{R})$.

A general approximation for functions $F \in \mathbb{F}_c(\mathbb{R})$ can be obtained by adopting parametric monotonic functions of the same type as suggested in [33] and [34], e.g., the (2,2)-rational function $p : [0,1] \longrightarrow [0,1]$ defined, for fixed but arbitrary $\beta_0, \beta_1 \geq 0$, by

$$p(t; \beta_0, \beta_1) = \frac{t^2 + \beta_0 t(1-t)}{1 + (\beta_0 + \beta_1 - 2)t(1-t)}, \; t \in [0,1] \tag{20}$$

The basic properties of p are that, for all $\beta_0, \beta_1 \geq 0$, $p(0; \beta_0, \beta_1) = 0$, $p(1; \beta_0, \beta_1) = 1$, its derivative is nonnegative (considering right derivative at $t = 0$ and left derivative at $t = 1$) and $p'(0; \beta_0, \beta_1) = \beta_0$, $p'(1; \beta_0, \beta_1) = \beta_1$. By changing the values of $\beta_0, \beta_1 \geq 0$, functions (20) generate an infinite number of monotonic increasing functions.

The "shape" functions $p(t; \beta_0, \beta_1)$ can be adopted to represent functions $F \in \mathbb{F}_c(\mathbb{R})$ "piecewise" on two decompositions of the intervals $[a_F, c_F]$ and $[d_F, b_F]$ into N_L subintervals $a_F = x_0^L < x_1^L < ... < x_{N_L}^L = c_F$ and N_R subintervals $d_F = x_0^R < x_1^R < ... < x_{N_R}^R = b_F$; at the extremal points of each subinterval $I_i^L = \left[x_{i-1}^L, x_i^L\right]$ and $I_j^R = \left[x_{j-1}^R, x_j^R\right]$ the values of F are fixed to the nondecreasing values F_i^L, $i = 0, 1, ..., N_L$ and F_j^R, $j = 0, 1, ..., N_R$ with $F_0^L = 0 < F_i^L < F_{N_L}^L = \frac{1}{2}$ (with $F_{i-1}^L \leq F_i^L$ for $i = 2, ..., N_L - 1$) and $F_0^R = \frac{1}{2} < F_j^R < F_{N_R}^R$ (with $F_{j-1}^R \leq F_j^R$ for $j = 2, ..., N_R - 1$).

Finally, $F \in \mathbb{F}_c(\mathbb{R})$, is constructed by choosing $N_L + N_R$ pairs of nonnegative parameters $(\beta_{0,i}^L, \beta_{1,i}^L)$, $(\beta_{0,j}^R, \beta_{1,j}^R)$ for all $i = 1, ..., N_L$ and $j = 1, ..., N_R$ (the slopes of F at the extremes of each subinterval I_i^L and I_j^R) and by setting

$$F(x) = \begin{cases} 0 & \text{if} \quad x < a_F \\ F_{i-1}^L + (F_i^L - F_{i-1}^L) p(\frac{x - x_{i-1}^L}{x_i^L - x_{i-1}^L}; \beta_{0,i}^L, \beta_{1,i}^L) & \text{if} \quad x_{i-1}^L \leq x < x_i^L \\ \frac{1}{2} & \text{if} \quad c_F \leq x \leq d_F \\ F_{j-1}^R + (F_j^R - F_{j-1}^R) p(\frac{x - x_{j-1}^R}{x_j^R - x_{j-1}^R}; \beta_{0,j}^R, \beta_{1,j}^R) & \text{if} \quad x_{j-1}^R < x \leq x_j^R \\ 1 & \text{if} \quad x > b_F \end{cases}$$

It is easy to check that the construction leads to functions $F \in \mathbb{F}_c(\mathbb{R})$.

For the case when $N_L = N_R = 1$, we have a simple construction where a single standardized function $p\left(t; \beta_0^L, \beta_1^L\right)$ is used to represent F on $[a_F, c_F]$ and another $p\left(t; \beta_0^R, \beta_1^R\right)$ to represent F on $[d_F, b_F]$

$$F(x) = \begin{cases} 0 & \text{if} \quad x < a_F \\ p(\frac{x-a_F}{c_F-a_F}; \beta_0^L, \beta_1^L) & \text{if} \quad a_F \leq x < c_F \\ \frac{1}{2} & \text{if} \quad c_F \leq x \leq d_F \\ \frac{1}{2}\left(1 + p(\frac{x-d_F}{b_F-d_F}; \beta_0^R, \beta_1^R)\right) & \text{if} \quad d_F < x \leq b_F \\ 1 & \text{if} \quad x > b_F \end{cases} .$$

This simple construction requires 8 parameters, i.e. the four values $a_F \leq c_F \leq d_F \leq b_F$ for the support and the core of the corresponding u_F and the four nonnegative parameters (β_0^L, β_1^L) and (β_0^R, β_1^R) used to fix the slopes of F at the points a_F, c_F and d_F, b_F, respectively.

Examples of functions F and corresponding u_F for different pairs (β_0^L, β_1^L) and (β_0^R, β_1^R) are pictured in figure 7, where $a_F = 1$, $b_F = 5$, $c_F = 2 + 0.5\,\text{rand}()$, $d_F = 3 + 0.5\,\text{rand}()$ and the βs are generated between 0 and 2.

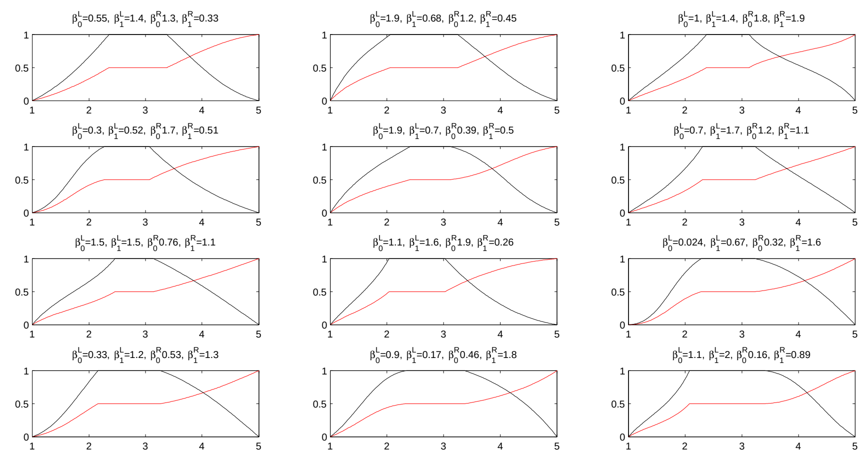

Fig. 7: Randomly generated $F \in \mathbb{F}_c(\mathbb{R})$ and corresponding fuzzy intervals u_F

In applications where we are interested to generate fuzzy numbers u_F with a single-valued core, without specifying its value *a-priori*, we can model the ACF by fixing the support $[a_F, b_F]$, $a_F < b_F$, and the two end-slope parameters $\beta_0^L = \beta_a \geq 0$, $\beta_1^R = \beta_b \geq 0$ so that

$$F(x) = \begin{cases} 0 & \text{if} \quad x < a_F \\ p(\frac{x-a_F}{b_F-a_F}; \beta_a, \beta_b) & \text{if} \quad a_F \leq x \leq b_F \\ 1 & \text{if} \quad x > b_F \end{cases} ; \tag{21}$$

the core of u_F can be computed simply by solving for the unique value of $x \in [a_F, b_F]$ that solves the equation $F(x) = \frac{1}{2}$, i.e. by solving the equation

$$q(t) = t^2 + \beta_a t(1-t) - \frac{1}{2} - \frac{1}{2}(\beta_a + \beta_b - 2)t(1-t) = 0$$

for the unique root $t_F \in]0, 1[$ (observe that $q(0) = -\frac{1}{2}$ and $q(1) = \frac{1}{2}$ and $q(t)$ is quadratic), then the core of u_F is obtained as $c_F = a_F + t_F(b_F - a_F)$.

Some examples for different values of β_a, β_b are given in figure 8, where $a_F = 1$, $b_F = 5$, $c_F = d_F$ (the core is a singleton) and β_a, β_b are generated between 0 and 5.

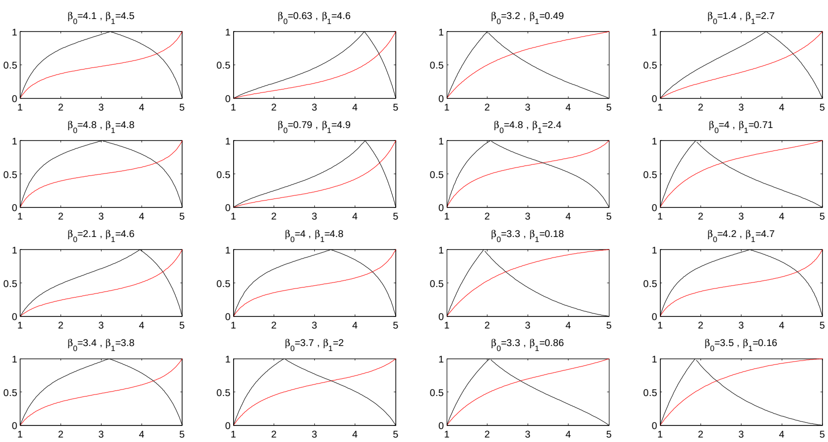

Fig. 8: Randomly generated $F \in \mathbb{F}_c(\mathbb{R})$ and corresponding fuzzy numbers u_F

A final interesting case with $\beta_a = \beta_b = \beta$ is in figure 9. Again, $a_F = 1$, $b_F = 5$, and β is generated randomly between 0 and 2.

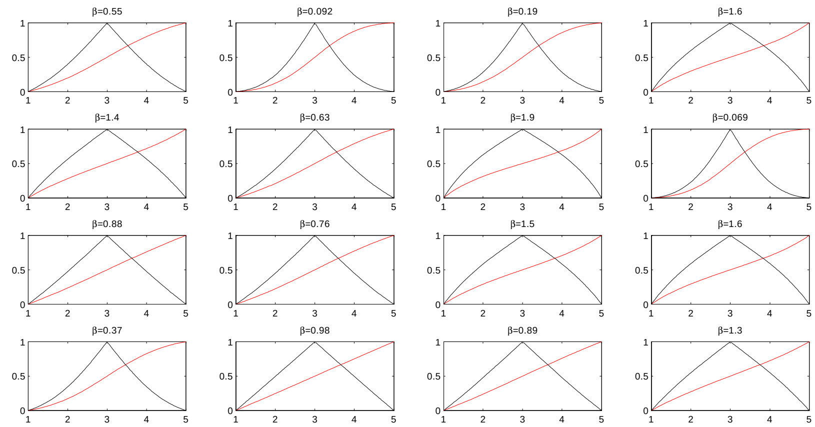

Fig. 9 : Randomly generated $F \in \mathbb{F}_c(\mathbb{R})$ and corresponding core-symmetric u_F

It is immediate to see that in the last case (if $\beta_a = \beta_b$ in (21)) the associated fuzzy number is symmetric with respect to the core $c_F = \frac{a_F + b_F}{2}$.

5 Conclusions and further research

We introduce the ACF-representation of fuzzy intervals, its parametric form and its properties; the ACF is based on some concepts shared with possibility theory.

We prove that the ACF can be uniquely defined for any fuzzy interval, we establish a relationship between ACF and quantile functions with a possibly statistical interpretation.

Further research involves some theoretical aspects about several topics:

- arithmetic operations: using an approach similar to probabilistic arithmetic, which is based on convolutions with density functions (as in [32] and [26], [20]), we will try to express fuzzy arithmetic in terms of AC functions, as we have done for scalar multiplication and addition in subsection 2.1. This approach has been extensively addressed by developing a very efficient software like the packages "distr" and "distrEx" in R language ([23]);
- membership estimation through observations (see for example [15] and [7]);
- possible metrics on ACFs that focus on useful topological structures (see for example [31] and [35]);

- ACF approximation through F-transform: the ACF-representation based on monotonic functions eases the search of approximation methods and algorithms (as in [3] and [4]);
- relationship between the probability distributions and membership functions ([30]).

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