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Quantile and expectile smoothing by F-transform

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Abstract

In this paper we illustrate the F-transform based on generalized fuzzy partitions as a tool for quantile and expectile smoothing. This allows to represent a time series in terms of a fuzzy-valued function whose level-cuts are modeled by F-transform and estimated by quantile or expectile regression. The proposed methodology is illustrated on several historical financial time series in order to highlight its strong properties .

1 Introduction

We will consider only the one-dimensional case. Given a continuous function $f : [a, b] \rightarrow \mathbb{R}$ and given a finite family of fuzzy sets (in particular fuzzy numbers) $\mathbb{A} = \{A_1, A_2, \dots, A_n\}$ forming a fuzzy partition of $[a, b]$, the F-transform produces a vector of real numbers $\mathbf{F} = (F_1, F_2, \dots, F_n)$ (called the *direct* F-transform) as a transformation $\mathbb{A} \rightarrow \mathbf{F}$ of the given family of fuzzy numbers. Each F_k is the minimizer of a weighted squared error (deviation) between the values $f(x)$ and F_k on the k -th subinterval of $[a, b]$. The direct F-transform $\mathbf{F}$ is then used to define the *inverse* F-transform function $\hat{f} : [a, b] \rightarrow \mathbb{R}$ and the main result is that $\hat{f}$ is an approximating function of f on $[a, b]$.

The key elements of the F-transform are clearly (a) the form of the weighting fuzzy numbers A_k and (b) how they are assembled to obtain the fuzzy partition of the interval $[a, b]$. In this work, we suggest two generalizations related to (a) and (b). With respect to point (a), we use a parameterized class of LR fuzzy numbers (see [20], [21]) to represent each A_k and obtain parametric partitions for the fuzzy transform; we will see that the proposed parameterized fuzzy partitions allow the possibility of estimating the shapes of fuzzy numbers to obtain a flexible F-transform with adaptability properties for approximation. With respect to point (b), we generalize fuzzy partitions in such a way that each weighting fuzzy number A_k "covers" a variable number of subintervals (the *bandwidth* of the partition) in the given decomposition of $[a, b]$.

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A preliminary version of the use of a parametric class of fuzzy numbers for the F-transform has been suggested in [18]; the idea of a generalized fuzzy partition has been presented in [19]. In this paper, we will illustrate the F-transform obtained on a generalized fuzzy partition and analyze its smoothing properties; we will also show a general procedure to determine the optimal bandwidth by the generalized cross validation (GCV) method or by controlling the total variation of the approximation (in particular, a finite taut-string variation). The F-transform setting appears to be a valid nonparametric smoothing methodology and its basic properties are similar, for example, to the well-known kernel or spline regression techniques.

The paper is organized into 4 sections: after the introduction we recall in section 2 the basic concepts about Fuzzy transform.

A preliminary version of the paper is in [6].

2 Basic elements about F-transform

We briefly recall the basic definitions and properties of the F-transform (see [10] for details).

A fuzzy partition for a given real compact interval $[a, b]$ is constructed by a decomposition $\mathbb{P} = \{a = x_1 < x_2 < \dots < x_n = b\}$ of $[a, b]$ into $n-1$ subintervals $[x_{k-1}, x_k]$, $k = 2, \dots, n$ and by a family $\mathbb{A} = \{A_1, A_2, \dots, A_n\}$ of n fuzzy numbers (the basic functions) identified by the membership values $A_1(x), A_2(x), \dots, A_n(x)$ for $x \in [a, b]$; the required properties (to complete this notation, we set $x_0 = a$ and $x_{n+1} = b$) are as follows:

1. each $A_k : [a, b] \longrightarrow [0, 1]$ is continuous with $A_k(x_k) = 1$, $A_k(x) = 0$ for $x \notin [x_{k-1}, x_{k+1}]$;
2. for $k = 2, 3, \dots, n-1$, A_k is increasing on $[x_{k-1}, x_k]$ and decreasing on $[x_k, x_{k+1}]$; A_1 is decreasing on $[a, x_2]$; A_n is increasing on $[x_{n-1}, b]$;
3. for all $x \in [a, b]$ the following partition-of-unity condition holds

$$\sum_{k=1}^n A_k(x) = 1.$$

Notation: We denote a fuzzy partition by the pair $(\mathbb{P}, \mathbb{A})$. The basic functions (fuzzy numbers) $A_1, A_2, \dots, A_n$ of $\mathbb{A}$ can be defined by $n-1$ increasing functions $L_2(x), \dots, L_n(x)$ as follows:

$$\begin{aligned} A_1(x) &= 1 - L_2(x) \text{ if } x \in [a, x_2], \\ A_k(x) &= \begin{cases} L_k(x) & \text{if } x \in [x_{k-1}, x_k] \\ 1 - L_{k+1}(x) & \text{if } x \in [x_k, x_{k+1}] \end{cases} \text{ for } k = 2, \dots, n-1, \\ A_n(x) &= L_n(x) \text{ if } x \in [x_{n-1}, b] \end{aligned} \quad (1)$$

where $L_k(x_{k-1}) = 0$ and $L_k(x_k) = 1$.

On each subinterval $]x_{k-1}, x_k[$ of the decomposition $\mathbb{P}$, only two basic functions $A_{k-1}(x)$ and $A_k(x)$ are non zero for $k = 2, \dots, n$.

To simplify the notation, given the n points $x_1 < x_2 < \dots < x_n$ of $\mathbb{P}$, we introduce the two additional points $x_0 = x_1$ and $x_{n+1} = x_n$ and we write, e.g., $A_k(x) \neq 0$ for all x in the open interval $]x_{k-1}, x_{k+1}[$, $k = 1, 2, \dots, n$.

Definition 2: (from [10]) Given a continuous function $f : [a, b] \longrightarrow \mathbb{R}$ and a fuzzy partition $(\mathbb{P}, \mathbb{A})$ of $[a, b]$, the direct fuzzy transform (F-transform) of f with respect to $(\mathbb{P}, \mathbb{A})$ is the n -tuple of real numbers $\mathbf{F} = (F_1, F_2, \dots, F_n)^T$ given by (notation $(\cdot)^T$ means transposition)

$$F_k = \frac{\int_a^b f(x) A_k(x) dx}{\int_a^b A_k(x) dx} = \frac{\int_{x_{k-1}}^{x_{k+1}} f(x) A_k(x) dx}{\int_{x_{k-1}}^{x_{k+1}} A_k(x) dx}, \quad k = 1, 2, \dots, n \quad (2)$$

Definition 3: (from [10]) Given the direct fuzzy transform $(F_1, F_2, \dots, F_n)^T$ of a continuous function $f : [a, b] \longrightarrow \mathbb{R}$ on a fuzzy partition $(\mathbb{P}, \mathbb{A})$, the inverse F-transform (iF -transform) is the continuous function $\widehat{f}_{\mathbf{F}} : [a, b] \longrightarrow \mathbb{R}$ given by

$$\widehat{f}_{\mathbf{F}}(x) = \sum_{k=1}^n F_k A_k(x) \text{ for } x \in [a, b]. \quad (3)$$

Remark 1: Note, by fixing the k -th subinterval of the decomposition $\mathbb{P}$, only $A_k(x)$ and $A_{k+1}(x)$ are nonzero for $x \in]x_k, x_{k+1}[$; more precisely, for each $k = 1, \dots, n-1$ and $i \neq k, k+1$, we have that $A_i(x) = 0$ for all $x \in [x_k, x_{k+1}]$. It follows that

$$\widehat{f}_{\mathbf{F}}(x) = F_k A_k(x) + F_{k+1} A_{k+1}(x) \quad \text{for } x \in [x_k, x_{k+1}], \quad k = 1, \dots, n-1$$

and, taking (1) into account,

$$\widehat{f}_{\mathbf{F}}(x) = F_k + L_{k+1}(x)(F_{k+1} - F_k) \quad \text{for } x \in [x_k, x_{k+1}], \quad k = 1, \dots, n-1. \quad (4)$$

The following approximation property ([10]) is one of the fundamentals of the F-transform setting.

Theorem 1: ([10]) If $f : [a, b] \longrightarrow \mathbb{R}$ is a continuous function then, for any positive real ε , there exists a fuzzy partition $(\mathbb{P}_{\varepsilon}, \mathbb{A}_{\varepsilon})$ such that the associated F-transform $\mathbf{F}_{\varepsilon} = (F_{1,\varepsilon}, F_{2,\varepsilon}, \dots, F_{n_{\varepsilon},\varepsilon})^T$ and the corresponding iF -transform $\widehat{f}_{\mathbf{F}_{\varepsilon}} : [a, b] \longrightarrow \mathbb{R}$ satisfy

$$|f(x) - \widehat{f}_{\mathbf{F}_{\varepsilon}}(x)| < \varepsilon \text{ for all } x \in [a, b].$$

To define a fuzzy partition $(\mathbb{P}, \mathbb{A})$, we can use a general decomposition $\mathbb{P} = \{a = x_1 < x_2 < \dots < x_n = b\}$ of $[a, b]$ into $n-1$ subintervals $[x_k, x_{k+1}]$ and a family $\mathbb{A} = \{A_1, A_2, \dots, A_n\}$ of n fuzzy numbers on $[a, b]$.

Denote $h_1 = 0$, $h_k = x_k - x_{k-1}$, $k = 2, \dots, n$, $h_{n+1} = 0$ and $h = \max\{h_k | k = 2, \dots, n\}$; if the decomposition $\mathbb{P}$ is uniform, we have $x_k = a + (k-1)h$, where $h = \frac{b-a}{n-1}$.

For a given continuous function $f : [a, b] \longrightarrow \mathbb{R}$, denote also, for $k = 1, \dots, n-1$,

$$M_{k,\mathbb{P}}(f) = \max\{|f(x) - f(t)| \mid \forall x \in [x_k, x_{k+1}] \text{ and } \forall t \in [x_{k-1}, x_{k+2}]\}.$$

Because $A_{k-1}(x) + A_k(x) = 1$ for $x \in [x_{k-1}, x_k]$, we have $\int_{x_{k-1}}^{x_k} [A_{k-1}(x) + A_k(x)]dx = h_k$. Denote

$$I_k = \int_a^b A_k(x)dx = \begin{cases} \int_{x_1}^{x_2} A_k(x)dx & k = 1 \\ \int_{x_{k-1}}^{x_{k+1}} A_k(x)dx & k = 2, \dots, n-1 \\ \int_{x_{n-1}}^b A_k(x)dx & k = n \end{cases} \quad (5)$$

With notation (5), the direct F-transform is

$$F_k = \begin{cases} \frac{1}{I_1} \int_{x_1}^{x_2} f(x) A_1(x) dx & k = 1 \\ \frac{1}{I_k} \int_{x_{k-1}}^{x_{k+1}} f(x) A_k(x) dx & k = 2, \dots, n-1 \\ \frac{1}{I_n} \int_{x_{n-1}}^b f(x) A_n(x) dx & k = n \end{cases} \quad (6)$$

The basic properties of the F-transform (see section 3 of [10]) are as follows.

Proposition 1: *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function and let $\mathbf{F} = (F_1, F_2, \dots, F_n)^T$ be its F-transform with respect to a given partition $(\mathbb{P}, \mathbb{A})$. Then*

(i) *For all $x \in [x_k, x_{k+1}]$ and $k = 1, 2, \dots, n-1$, we have*

$$\begin{aligned} |f(x) - F_k| &\leq M_{k,\mathbb{P}}(f) \\ |f(x) - F_{k+1}| &\leq M_{k,\mathbb{P}}(f). \end{aligned}$$

(ii) *For all $k = 1, \dots, n$, there exist $a \leq c_1'' \leq x_2$, $x_{k-1} \leq c_k' \leq c_k'' \leq x_{k+1}$ ($k = 2, \dots, n-1$) and $x_{n-1} \leq c_n' \leq b$ such that*

$$\begin{aligned} F_1 &= \frac{1}{I_1} \int_a^{c_1''} f(x) A_1(x) dx, \quad F_n = \frac{1}{I_n} \int_{c_n'}^b f(x) A_n(x) dx \\ F_k &= \frac{1}{I_k} \int_{c_k'}^{c_k''} f(x) A_k(x) dx \text{ for } k = 2, \dots, n-1. \end{aligned}$$

(iii) For all $k = 1, \dots, n-1$, we have

$$\int_{x_{k-1}}^{x_{k+1}} |f(x) - \widehat{f}_{\mathbf{F}}(x)| dx \leq M_{k,\mathbb{P}}(f)(x_{k+1} - x_{k-1}).$$

(iv) Each F_k is a minimizer of the function $\phi_k(y) = \int_a^b (f(x) - y)^2 A_k(x) dx$.

(v) If f is twice continuously differentiable and if the basic functions $A_k \in \mathbb{A}$ are such that $I_1 = \frac{x_2 - x_1}{2}$, $I_k = \frac{x_{k+1} - x_{k-1}}{2}$ for $k = 2, \dots, n-1$ and $I_n = \frac{x_n - x_{n-1}}{2}$, then

$$F_k = f(x_k) + O(\widehat{h}_k^2) \quad \text{where} \quad \widehat{h}_k = \max\{h_k, h_{k+1}\}$$

for all $k = 1, \dots, n$.

The F-transform has two properties that are useful for its interpretation. The first states an invariance property of the integral.

Theorem 2: Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be two functions and let $\widehat{f}, \widehat{g} : [a, b] \longrightarrow \mathbb{R}$ be the iF-transforms of f and g with respect to the same fuzzy partition $(\mathbb{P}, \mathbb{A})$. Then

$$\int_a^b \widehat{f}(x) g(x) dx = \int_a^b f(x) \widehat{g}(x) dx; \quad (7)$$

in particular,

$$\int_a^b \widehat{f}(x) dx = \int_a^b f(x) dx. \quad (8)$$

The second property shows that each component F_k of the F-transform can be expressed as the value $f(\xi_k)$ of f at a point $\xi_k \in [x_{k-1}, x_{k+1}]$; consequently, the iF-transform can be expressed as a linear combination of the basic functions A_k with coefficients given by values of f at points ξ_k in its domain.

Theorem 3: Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function and let $\widehat{f} : [a, b] \longrightarrow \mathbb{R}$ be its iF-transform with respect to a fuzzy partition $(\mathbb{P}, \mathbb{A})$. Then there exist fixed points $\xi_k \in [x_{k-1}, x_{k+1}]$ such that $F_k = f(\xi_k)$ and

$$\widehat{f}(x) = \sum_{k=1}^n F_k A_k(x) = \sum_{k=1}^n f(\xi_k) A_k(x). \quad (9)$$

In particular,

$$F_k = \widehat{f}(x_k) = f(\xi_k) \quad \text{for all } k = 1, \dots, n. \quad (10)$$

The direct and inverse F-transforms have been defined in [10] for the discrete case as follows. A compact interval $[a, b]$ is given, and a fuzzy partition $(\mathbb{P}, \mathbb{A})$ of $[a, b]$ is selected with the usual notation. Given m distinct points $t_j \in [a, b]$,

$j = 1, \dots, m$, such that each set $T_k = \{t_j | A_k(t_j) > 0\}$, $k = 1, \dots, n$, is nonempty (assuming $t_1 < t_2 < \dots < t_m$, we say in this case that $\mathbb{T} = \{t_1, t_2, \dots, t_m\}$ is *sufficiently dense* with respect to $(\mathbb{P}, \mathbb{A})$), the *discrete direct F-transform* of a function $f : [a, b] \longrightarrow \mathbb{R}$ is defined by

$$F_k = \frac{1}{s_k} \sum_{j=1}^m f(t_j) A_k(t_j) \text{ for } k = 1, \dots, n \quad (11)$$

$$s_k = \sum_{j=1}^m A_k(t_j).$$

To simplify the notation, the corresponding iF-transform function will be denoted, instead of by $\hat{f}_{\mathbf{F}}(t)$, by

$$\hat{f}(t) = \sum_{k=1}^n F_k A_k(t) \text{ for } t \in [a, b]. \quad (12)$$

We will also use the m -dimensional vector $\hat{\mathbf{f}} = (\hat{f}_1, \dots, \hat{f}_m)$ with components $\hat{f}_j = \hat{f}(t_j)$ obtained by evaluating the iF-transform function $\hat{f} = \sum_{k=1}^n F_k A_k$ at each point t_j , for $j = 1, \dots, m$.

Theorem 2 has an analog version in the discrete F-transform setting.

Theorem 4: *Let $(\mathbb{P}, \mathbb{A})$ be a fuzzy partition of $[a, b]$; let m points $t_j \in [a, b]$ be given such that $\mathbb{T} = \{t_1, t_2, \dots, t_m\}$ is sufficiently dense with respect to $(\mathbb{P}, \mathbb{A})$; and let $f : [a, b] \longrightarrow \mathbb{R}$, $g : [a, b] \longrightarrow \mathbb{R}$ be two functions with discrete iF-transforms $\hat{f}(t)$ and $\hat{g}(t)$, respectively. Then*

$$\sum_{j=1}^m \hat{f}(t_j) g(t_j) = \sum_{j=1}^m f(t_j) \hat{g}(t_j) \quad (13)$$

and, in particular,

$$\sum_{j=1}^m \hat{f}(t_j) = \sum_{j=1}^m f(t_j). \quad (14)$$

The basic functions $A_1, A_2, \dots, A_n$ of the fuzzy partition $(\mathbb{P}, \mathbb{A})$ are defined by $n - 1$ increasing functions $L_2, \dots, L_n$ as in (1). As suggested in [18], each function $L_k : [0, 1] \longrightarrow [0, 1]$ ($k = 2, \dots, n$) can be modeled in parametric form as it is detailed in [22].

3 Use of the F-transform in expectile smoothing

In order to investigate the role of F-transform in expectile smoothing, let's first introduce some basic facts on quantile and expectile regression and autoregression, a recent interesting field in nonparametric regression (see e.g. [16], [7] and the references therein).

Consider a real-valued random variable ξ ; a given r -quantile $\xi(r)$ is defined by the property that the probability that an observation is less than $\xi(r)$ is r , with $r \in]0, 1[$

$$Prob(\xi \leq \xi(r)) = r.$$

Given a set of T observations x_t , $t = 1, \dots, T$, the sample quantile $\bar{\xi}(r)$ can be obtained as the solution to minimize the function (with respect to m)

$$S_r(m) = \sum_{t=1}^T \rho_r(x_t - m)$$

where $\rho_r(y)$ is the *check function* (see e.g. [16]) defined for quantiles as

$$\rho_r(y) = (r - I(y < 0))y$$

and the function $I(\cdot)$ is defined by

$$I(y < 0) = \begin{cases} 1 & \text{if } y < 0 \\ 0 & \text{otherwise} \end{cases}.$$

We can write

$$S_r(m) = (r - 1) \sum_{x_t < m} (x_t - m) + r \sum_{x_t > m} (x_t - m)$$

and it is immediate to see that $S_r(m) \geq 0$ for all real m .

If $r = \frac{1}{2}$, then the r -quantile gives the median m_e of the (empirical) distribution of ξ , i.e. the minimizer of the functional

$$m_e = \arg \min_m S_{1/2}(m) = \sum_{t=1}^T |x_t - m|.$$

The expectiles are defined in a similar way as for quantiles, in particular by using the mean instead of the median.

The population expectiles $\tilde{\mu}(\omega)$, for $\omega \in]0, 1[$, are defined by tail expectations rather than tail probabilities. For a given value of $\omega \in]0, 1[$, the sample expectile $\tilde{\mu}(\omega)$ is obtained by minimizing

$$S_\omega(\mu) = \sum_{t=1}^T \rho_\omega(x_t - \mu), \quad \omega \in]0, 1[$$

where the check function is now given by

$$\rho_\omega(y) = y^2 |\omega - I(y < 0)|.$$

This is equivalent to

$$S_\omega(\mu) = \sum_{\substack{t=1 \\ x_t < \mu}}^T (1 - \omega) (x_t - \mu)^2 + \sum_{\substack{t=1 \\ x_t > \mu}}^T \omega (x_t - \mu)^2.$$

If $\omega = \frac{1}{2}$ we obtain the mean value μ_e of the observations

$$\begin{aligned}\mu_e &= \arg \min_{\mu} S_{\frac{1}{2}}(\mu) = \frac{1}{2} \sum_{t=1}^T (x_t - \mu)^2 \\ \mu_e &= \frac{1}{T} \sum_{t=1}^T x_t\end{aligned}$$

The ordinary least squares (OLS) estimation of μ is obtained by minimizing

$$S_{OLS} = \sum_{t=1}^T (x_t - \mu)^2;$$

when an asymmetry parameter $\omega \in]0, 1[$ is chosen, then the least asymmetrical weighted squares (LAWS) function

$$S_{LAWS}^{(\omega)}(\mu) = \sum_{t=1}^T w_{\omega}(t) (x_t - \mu)^2 \quad (15)$$

is minimized, where the weights $w_{\omega}(t)$ are

$$w_{\omega}(t) = \begin{cases} \omega & \text{if } x_t > \mu \\ 1 - \omega & \text{if } x_t \leq \mu \end{cases}$$

The value $\mu = \mu(\omega)$ (depending on ω) is the population expectile for different values of the asymmetry parameter $\omega \in]0, 1[$. The model in (15) can be fitted by applying iteratively a weighted least squares minimization (see e.g. [23]).

Iterated LAWS Algorithm:

1. Chose a value $\omega \in]0, 1[$ and start with estimated (initial) w_i^l , $l = 0$ (e.g. $w_t^0 = \frac{1}{2}$ for all $t = 1, \dots, T$);
2. Compute the minimizer μ^l of $S_{LAWS}^{(\omega)}(\mu) = \sum_{t=1}^T w_t^l (x_t - \mu)^2$ and compute new weights
$$w_t^{l+1} = \begin{cases} \omega & \text{if } x_t > \mu^l \\ 1 - \omega & \text{if } x_t \leq \mu^l \end{cases}$$

increase $l = l + 1$;
3. Continue iteratively with step 2 until the weights w_t^{l+1} become "stable" with respect to w_t^l (i.e. $w_t^{l+1} = w_t^l$ for all t).

Note that ordinary least squares is a special case of LAWS when $\omega = \frac{1}{2}$, and the solution of (15) in step 2 is given by

$$\mu^l = \sum_{t=1}^T w_t^l x_t \bigg/ \sum_{t=1}^T w_t^l.$$

The expectile F-transform, for a fixed generalized fuzzy r -partition $(\mathbb{P}, \mathbb{A}^{(r)})$ and for a given value of $\omega \in]0, 1]$, can be defined, according to the expectiles setting described above, to be the minimizer of the following operators, for $k = 1, \dots, n$,

$$\begin{aligned} \Phi_{k,\omega}(F) &= \int_a^b w_\omega(x) (f(x) - F)^2 A_k^{(r)}(x) dx \\ &\text{where} \\ w_\omega(x) &= \begin{cases} \omega & \text{if } f(x) \leq F \\ 1 - \omega & \text{if } f(x) > F \end{cases}. \end{aligned}$$

Remark that if $\omega = 0.5$ the minimization of $\Phi_{k,0.5}(F)$ with respect to F gives the usual F-transform component $F_{k,0.5}$ in Definition 3.

According to the fact that, corresponding to $\omega > \frac{1}{2}$ we obtain a value of F greater than $F_{k,0.5}$, and that, corresponding to $\omega < \frac{1}{2}$ we obtain a value of F less than $F_{k,0.5}$, we suggest the following procedure to obtain a fuzzy-valued version of the direct F-transform:

Choose $\alpha \in]0, 1]$ and consider the two operators $\Phi_{k,\alpha}^-(F)$ and $\Phi_{k,\alpha}^+(F)$, defined by

$$\begin{aligned} \Phi_{k,\alpha}^-(F) &= \int_a^b w_\alpha^-(x) (f(x) - F)^2 A_k^{(r)}(x) dx \\ &\text{where} \\ w_\alpha^-(x) &= \begin{cases} \frac{\alpha}{2} & \text{if } f(x) \leq F \\ 1 - \frac{\alpha}{2} & \text{if } f(x) > F \end{cases} \end{aligned}$$

and

$$\begin{aligned} \Phi_{k,\alpha}^+(F) &= \int_a^b w_\alpha^+(x) (f(x) - F)^2 A_k^{(r)}(x) dx \\ &\text{where} \\ w_\alpha^+(x) &= \begin{cases} 1 - \frac{\alpha}{2} & \text{if } f(x) \leq F \\ \frac{\alpha}{2} & \text{if } f(x) > F \end{cases}. \end{aligned}$$

If $\alpha = 1$ we obtain $\Phi_{k,0.5}(F)$.

The minimization of $\Phi_{k,\alpha}^-(F)$ and $\Phi_{k,\alpha}^+(F)$ produces, respectively $F_{k,\alpha}^-$ and $F_{k,\alpha}^+$ so that $[F_{k,\alpha}^-, F_{k,\alpha}^+]$ is the α -cut of F_k .

As a consequence, the iF-transform of f is fuzzified by:

$$\widehat{f}(x) = \frac{1}{r} \sum_{k=1}^n F_k A_k^{(r)}(x)$$

with the corresponding α -cuts expressed as a linear combination of intervals $[F_{k,\alpha}^-, F_{k,\alpha}^+]$:

$$\begin{aligned} [\widehat{f}(x)]_\alpha &= [\widehat{f}_\alpha^-(x), \widehat{f}_\alpha^+(x)]_\alpha \\ &= \frac{1}{r} \sum_{k=1}^n [F_{k,\alpha}^-, F_{k,\alpha}^+] A_k^{(r)}(x) \end{aligned} \quad (16)$$

When $\alpha = 1$ we obtain the standard F -transform and the corresponding iF -transform.

The discrete case can be handled in a similar way as for the standard discrete F-transform. The expectiles, in the discrete case, are obtained by minimizing the following functions:

$$\Phi_{k,\alpha}^-(F) = \sum_{k=1}^m w_\alpha^-(t_i) (f(t_i) - F)^2 A_k^{(r)}(t_i) \quad (17)$$

where

$$w_\alpha^-(t_i) = \begin{cases} \frac{\alpha}{2} & \text{if } f(t_i) \leq F \\ 1 - \frac{\alpha}{2} & \text{if } f(t_i) > F \end{cases}$$

and

$$\Phi_{k,\alpha}^+(F) = \sum_{k=1}^m w_\alpha^+(t_i) (f(t_i) - F)^2 A_k^{(r)}(t_i) \quad (18)$$

where

$$w_\alpha^+(t_i) = \begin{cases} 1 - \frac{\alpha}{2} & \text{if } f(t_i) \leq F \\ \frac{\alpha}{2} & \text{if } f(t_i) > F \end{cases}$$

Consider that, for fixed values of $w_\alpha^\pm(t_i)$, the minimizers $F_{k,\alpha}^-$ and $F_{k,\alpha}^+$ of (17) and (18) are obtained, respectively, for $k = 1, \dots, n$, by

$$F_{k,\alpha}^- = \frac{\sum_{k=1}^m w_\alpha^-(t_i) f(t_i) A_k^{(r)}(t_i)}{\sum_{k=1}^m w_\alpha^-(t_i) A_k^{(r)}(t_i)}$$

and

$$F_{k,\alpha}^+ = \frac{\sum_{i=1}^m w_{\alpha}^+(t_i) f(t_i) A_k^{(r)}(t_i)}{\sum_{i=1}^m w_{\alpha}^+(t_i) A_k^{(r)}(t_i)}.$$

An iterative procedure can be easily designed, similar to the Iterated LAWS Algorithm as described above.

Remark that if $\alpha = \frac{1}{2}$, then we have $w_{\alpha}^-(t_i) = w_{\alpha}^+(t_i) = \frac{1}{2}$ for all $i = 1, \dots, m$ and we obtain exactly the standard discrete F-transform of f based on the observations $f(t_i)$ at the points $t_1, t_2, \dots, t_m$, assumed to be sufficiently dense with respect to the given fuzzy r -partition $(\mathbb{P}, \mathbb{A}^{(r)})$.

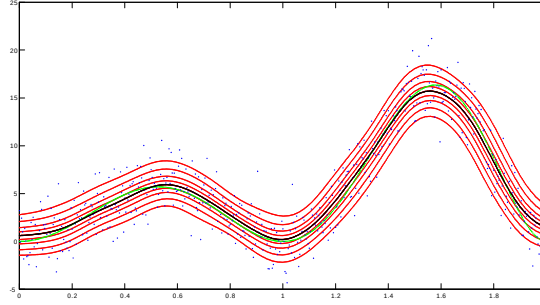


Figure 1. α -cuts of a fuzzy-valued function
by F-transform ($m = 501$, $n = 101$, $r = 6$) and
 $\alpha = 0.01, 0.25, 0.5, 0.75, 1.0$

In figure 2 we illustrate the expectile smoothing to the same simulated example as in [22] (Section 4.3, example 1): $f(t_i) = 5e^{-0.5t_i^2} \sin^2(\pi t_i) + 2z_i$, $t_i \in [0, 2]$, $i = 1, \dots, m$, where $z_i \in N(0, 1)$.

The data are represented by points and 9 curves are generated, corresponding to the values of $\alpha = 0.01, 0.25, 0.5, 0.75, 1.0$; it is to be remarked that for any value of $\alpha \in]0, 1]$ we can obtain the α -cut $[F_{k,\alpha}^-, F_{k,\alpha}^+]$ of F_k , $k = 1, 2, \dots, n$. The curves are then constructed by inverse F-transform, equation (16).

4 Application to hystorical financial time series

In order to show how the F-transform can be used for expectile smoothing, we apply the proposed estimation on some time series. In all the cases, the number n of subintervals in the fuzzy partition $(\mathbb{P}, \mathbb{A}^{(r)})$ is approximately $\frac{m}{5}$ and the bandwidth r is estimated by generalized cross validation as in [22]. In all cases, for simplicity, the basic functions $A_k(x)$, defined on the intervals $[x_{k-r}, x_{k+r}]$,

are obtained by translating and rescaling the same symmetric triangular fuzzy number T_0 , defined on $[-1, 1]$ and centered at the origin, with membership

$$T_0(t) = \begin{cases} 1 + t & \text{if } t \in [-1, 0] \\ 1 - t & \text{if } t \in [0, 1] \\ 0 & \text{otherwise} \end{cases}.$$

We consider three hystorical daily prices series with different properties related to market behaviour. The time period is from October, 9th 2007 to October, 8th 2015 and it follows that $m = 2016$ in the three cases.

The first time series contains the silver prices in US dollars. The price is set once a day by three London Bullion Market Association (LBMA) market makers that listen to customers' purpose as buyers or sellers: the silver fixing price is then set by collating bids and offers until the supply and demand are matched.

In general, commodities likes silver are considered an high volatility asset class but a small amount diversifies well risks associated with stocks, bonds and cash.

The second time series is about the wellknown Apple stock that has an high volatility in the short term.as all the stocks.

The third historical data series contains the S&P 500 that is probably the most accurate quantifier of the U.S. economy, measuring the cumulative float-adjusted market capitalization of 500 of the nation's largest corporations; due to its definition it is considered a low volatility stock.

Figures 3 and 3a represent the observations and the α -cuts obtained by the repeated application of F-transform (17), (18) for the indicated values of $\alpha \in]0, 1]$.

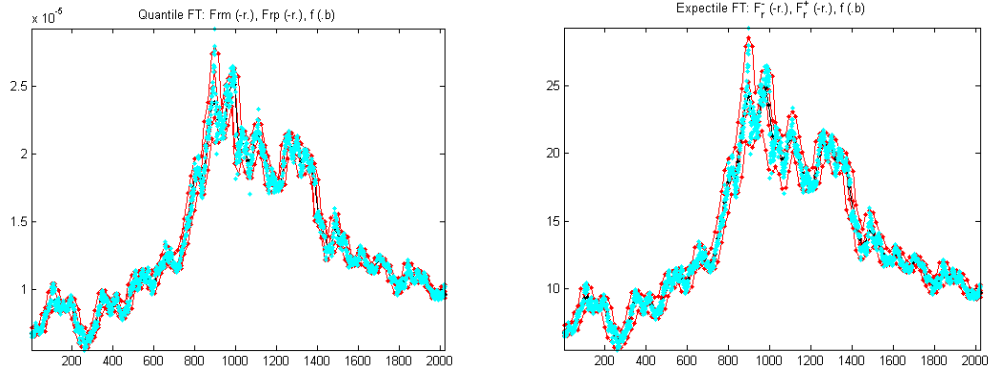


Figure 2: comparison between quantile and expectile smoothing

As a second collection of data we use 5 years of the FTSE 100, that is a share daily index of the 100 companies listed on the London Stock Exchange

with the highest market capitalization. It is one of the most widely used stock indices and is seen as an indicator of business prosperity. The $m = 1264$ cover the period from 1 June 2007 to 31 May 2012.

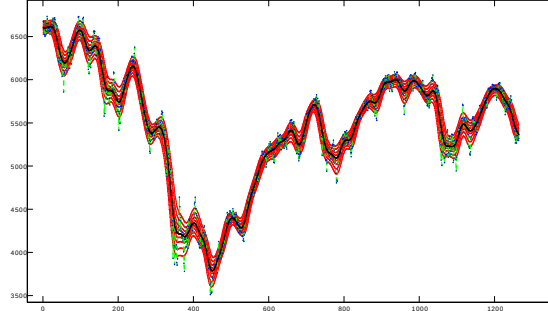


Figure 3. α -cuts of a fuzzy-valued function
by F-transform ($m = 1264$, $n = 250$, $r = 3$) and
 $\alpha = 0.01, 0.25, 0.5, 0.75, 1.0$

The fuzzy values (16) can be better distinguished in figure 4a.

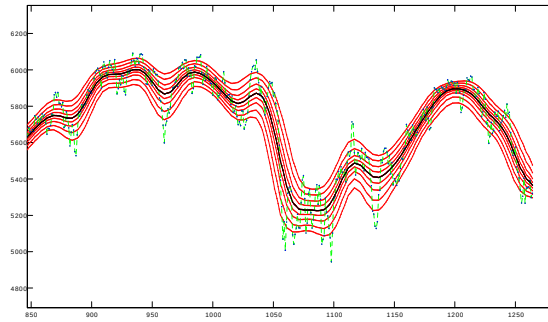


Figure 3a. Zoom of figure 4.

The third series of data considers the traded volumes of FTSE 100 above.

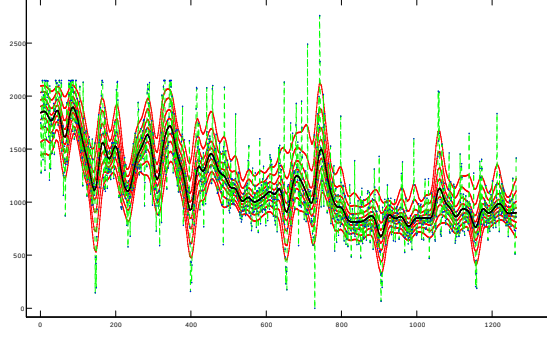


Figure 4. α -cuts of a fuzzy-valued function
by F-transform ($m = 1264$, $n = 250$, $r = 3$) and
 $\alpha = 0.01, 0.25, 0.5, 0.75, 1.0$

The last time series is the daily ITRAXX Europe index, with $m = 1380$ data from 16/6/2004 to 01/10/2009.

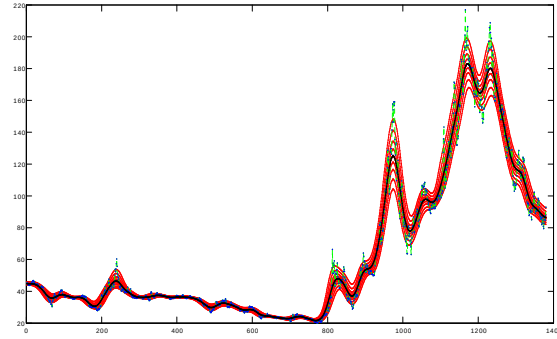


Figure 5. α -cuts of a fuzzy-valued function
by F-transform ($m = 1380$, $n = 276$, $r = 4$) and
 $\alpha = 0.01, 0.25, 0.5, 0.75, 1.0$

5 Conclusions

The introduced smoothing technique may represent a good alternative to the most popular ones because the algorithmic implementation is simple and the integral is preserved.

Expectile approach is probably more interesting because the related operator is differentiable while for the quantile this is not true.

Using an appropriate (generalized) fuzzy partition, the α -cuts $[F_{k,\alpha}^-, F_{k,\alpha}^+]$ of F_k have the same smoothing property inherited from F-transform, with a "degree of smoothness" depending on the bandwidth of the partition.

The preliminary results encourage to further work in the study and applications of F-transform as a tool to obtain a fuzzy-valued interpretation of a time series. In particular, we are interested to better understand the connections between the fuzzy partition $(\mathbb{P}, \mathbb{A}^{(r)})$ (the number and form of the basic functions, the length of the bandwidth,...) and the properties of the estimated fuzzy-valued (inverse) F-transform.

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