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"ROBUST UNBOUNDED CHAOTIC ATTRACTORS IN 1D DISCONTINUOUS MAPS"

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Robust unbounded chaotic attractors in 1D discontinuous maps

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Abstract

In this paper we prove the existence of *full measure unbounded chaotic attractors* which are persistent under parameter perturbation (also called robust). We show that this occurs in a discontinuous piecewise smooth one-dimensional map f, belonging to the family known as Nordmark's map. To prove the result we extend the properties of a full shift on a finite or infinite number of symbols to a map, here called Baker-like map with infinitely many branches, defined as a map of the interval $I = [0, 1]$ into itself with infinitely branches due to expanding functions with range I except at most the rightmost one. The proposed example is studied by using the first return map in I, which we prove to be chaotic in I making use of the border collision bifurcations curves of basic cycles. This leads to a robust unbounded chaotic attractor, the interval $(-\infty, 1]$, for the map f.

Kyewords. Unbounded chaotic attractors, Robust full measure chaotic attractors, Piecewise smooth systems, Full shift maps, Border collision bifurcations

1 Introduction

The study of the properties of one-dimensional discrete dynamical systems is mainly performed considering a function which maps a compact interval into itself. At the present time there are many works dealing with such systems, which consider both continuous and discontinuous maps (see e.g. [16], [7], [17], [24]). The various definitions of attractor given in the current literature refer almost all to compact sets ([24], [13], [26]). Moreover, the interest is often focussed on chaotic attractors, which in one-dimensional maps are cyclic or acyclic chaotic intervals, and thus bounded invariant sets in which the boundaries are given by the images of critical points (see [2]). In particular, in such cases the contact of the invariant set with the basin's boundary leads to a change in the dynamics. A typical example is the logistic map $T(x) = \mu x(1-x)$, at $\mu = 4$ a chaotic interval exists (not attracting) and for $\mu > 4$ mainly divergent dynamics occur (although an invariant chaotic set still exists in $[0, 1]$), and many examples can also be found in applied models.

However, there are several systems, also in applications, that lead to unbounded trajectories which are not diverging. This fact was emphasized for example in [6], and unbounded chaotic sets naturally arise in the iteration of maps with a vanishing denominator. For example, the existence of a "non bounded chaotic solution" in a one-dimensional map has been shown in [15] (see also [17] p.38). Further examples can be found in [5], where the related theory and properties were extended to two-dimensional maps.

In the references cited above, the existence of unbounded chaotic sets of full measure was proved on the basis of theoretical arguments, and in some cases even giving the closed analytical expression of such trajectories in terms of elementary algebraic and transcendental functions. The main analytical tool used in [5] to give the closed form solution of unbounded chaotic trajectories is related to a method based on the Schröder functional equation, described in [15] (see also [17] and the Appendix in [5]).

Clearly, a full measure unbounded chaotic set in a one-dimensional (1D for short) map must include periodic points which are dense in an unbounded interval. The basic characteristic of an unbounded and not

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diverging aperiodic trajectory is that it includes points arbitarily close to infinity which are then followed by points that come back (at finite distance). Such a property leads to difficulties in the numerical iterations of the system, since an overflow error occurs even if the numerically generated trajectory is not diverging. However, the technique applied in the present work, making use of the first return map of the system, ultimately leads to the study of a map in a compact interval, avoiding such kind of numerical problems and allowing suitable theoretical tools.

One more feature that is worth to mention is that the examples leading to a closed form solution of the trajectories refer to full measure unbounded chaotic sets which are not structurally stable.[1] That is, a small perturbation of the parameters in the system causes the destruction of the invariant chaotic set (at least in some characteristic features here mentioned, unbounded, chaotic, and full measure). However, an important property is the persistence of chaotic attractors (i.e. really attracting sets in the phase space) under parameter variations, also called *robust chaotic attractos*, following the definition given in [3]. One of the goals of the present work is to show that full measure unbounded chaotic attractors can exist which are also robust. In the proposed example, this property (existence of a full measure unbounded chaotic attractor) persists for parameters in a wide region of positive measure of the parameter space.

To get this result we need to prove the existence of full chaos in a discontinuous map of the interval $I = [0, 1]$ into itself with infinitely many discontinuity points, leading to a second goal. This subject is not new in the literature. The basic tools are related to a Baker-like map on the interval I, with two branches, which has been deeply studied since many years and is nowadays of common knowledge ([7]). That is, a map from I into itself with one discontinuity point, and expanding functions in the two partitions, having range I. It is also known as full shift on two symbols, and the same property of the map (of full chaos in I) also holds in a full shift on any finite number of symbols. That is, a map from I into itself with a finite number of discontinuity points, and in all the partitions expanding functions have range I.

The results have been extended to shifts on an infinite number of symbols in [27], and thus to a map with infinitely many branches (infinitely many discontinuity points) when all the infinite branches of the map have full range I, let us call it a *Baker map with infinitely many branches.* This result was used in [12], and it is used also in our study. In fact, we shall prove that the map may be reduced to this kind at the border collision bifurcations of basic cycles (as we shall clarify in the next sections). However, as it is common to occur, such bifurcation values belong to a set of zero measure in the parameter space, thus the unbounded chaotic attractor proved in this way cannot be called robust. In order to prove that it is really persistent, we need to extend the result on Baker maps with infinitely many branches (based on [27]) also to maps in which one branch is not of full range $[0, 1]$. This extension to what we call a *Baker-like map with infinitely many branches*, is necessary to reach the goal of proving the persistence of full measure unbounded chaotic attractors.

The plan of the paper is as follows. In Sec.2 we introduce a discontinuous map defined in two partitions belonging to the family proposed by Nordmark ([18], [19]), with a linear branch in the left side of $x = 0$ and an hyperbolic branch in $x > 0$. The branch on the right side has a vertical asymptote at $x = 0$. We prove that the dynamic of the system in the interesting parameter range can be studied by use of a suitable first return map, which can be analytically described. This return map of the interval $I = [0, 1]$ into itself has infinitely many discontinuity points ξ_j which are accumulating to $x = 0$. All the branches are defined by expanding functions having range I except at most the rightmost one, defined in a given interval $[\xi, 1]$, whose range can be any interval $[0, y]$ with $0 < y \leq 1$. That is, a Baker-like map with infinitely many discontinuity points in our definition, given in Sec.3, where we prove that it is chaotic in $[0, 1]$ by using the standard tools, that is, proving that transitivity occurs in I, that periodic points are dense and there is sensitivity with respect to the initial conditions. In Sec.4 we return to the proposed example where we use the border collision bifurcation curves together with the fold bifurcation curves of basic cycles to show that the first return map is a Baker-like map with infinitely many discontinuity points for parameters belonging to a wide region of the parameter space, of positive measure. Thus proving that the interval $(-\infty, 1]$ is a robust unbounded chaotic attractor for the map f. Sec.5 concludes.

[1] As it occurs, for example, in the logistic map $T(x) = 4x(1 - x)$.

2 One-dimensional discontinuous piecewise-smooth map

The 1D discontinuous map which we are interested in comes from the applied context. Recent applications in engineering lead to piecewise smooth systems (see e.g. [8]) among which much attention has been given to the system proposed by Nordmark ([18], [19]), defined as follow:

$$x \longmapsto f_\mu(x) = \begin{cases} f_L(x) = ax + \mu & \text{if} \quad x \le 0 \\ f_R(x) = bx^{-\gamma} + \mu & \text{if} \quad x > 0 \end{cases} \tag{1}$$

and mainly considered for $\gamma < 0$ and $\mu > 0$. Recently, the discontinuos case occurring for $\gamma > 0$ has been also investigated. It was first considered in [20], then in [21] for the particular case $\gamma = 0.5$, evidencing the existence regions of stable basic cycles, and in [14] a complete investigation for any $\gamma > 0$ has been performed. In particular, in the present work we are interested in the following parameter ranges:

$$0 < a \le 1, \ b < -1, \ \gamma > 0 \tag{2}$$

Regarding the parameter μ, without loss of generality it can be fixed at $\mu = 1$. In fact, for any $\mu > 0$ the transformation $(x, a, b, \mu) \longrightarrow (x\mu^{-1}, a, b\mu^{-\gamma-1}, 1)$ leads from (1) to the map

$$x \longmapsto f(x) = \begin{cases} f_L(x) = ax + 1 & \text{if} \quad x \le 0 \\ f_R(x) = \frac{b}{x^\gamma} + 1 & \text{if} \quad x > 0 \end{cases} \tag{3}$$

We recall that by using the symbolic notation based on the letters L and R corresponding to the two partitions $I_L = (-\infty, 0]$, $I_R = (0, +\infty)$, respectively, we may associate to each trajectory its itinerary by using the letter L when a point belongs to the partition I_L (L side for short) and R when a point belongs to the partition I_R (R side for short). A cycle can be represented by its finite symbolic sequence. For example, a cycle with symbolic sequence RL^n corresponds to a cycle having one periodic point on the right partition and n on the left one. Such cycles, or those with symbolic sequence LR^n, are called basic cycles (or maximal cycles, or principal cycles, see [8], [1], [9]). In the case of a piecewise smooth discontinuous map, it is common to be faced also with non smooth bifurcations, called border collision bifurcations (BCB for short). This term is here used to denote a periodic point of a cycle which is colliding with the discontinuity point $x = 0$ from the L side.

The properties of the dynamic behaviors in the considered region for the parameters given in (2) depend on the straight line on the L side having slope smaller that 1 and on the rank-1 preimage of the origin on the R side $O_R^{-1} = (-b)^{\frac{1}{\gamma}}$ which is larger than 1. Since no fixed point exists in the L side, any point on the left side has an increasing sequence reaching the right side in a point ≤ 1. At the same time, any point on the right side larger than 1 is mapped below 1 in one iteration. Thus we can consider the interval $(-\infty, 1]$ (range of the map f). Moreover, for $b < -1$ any point in $(0, 1]$ is mapped on the L side in one iteration. That is, in the itinerary of any trajectory the symbol R is necessarily followed by L (i.e. at least one L). Thus the only possible basic cycles are those with the symbolic sequence RL^n, and they all exist for any $n \ge 1$, in suitable parameter ranges. Indeed, let $0 < x_0 < 1$ be a point on the right side, then when b is small enough we have $f_R(x_0) << 0$, and it takes many iterations by f_L for the trajectory in order to reach the right side again.

To rigorously prove the dynamic properties of a map, it comes quite often useful to consider the first return map in a suitable interval (some recent examples in discontinuous maps can be found in [10], [11]). In the case of map f in (3), we can consider the first return map of f in the interval $I = [0, 1]$, whose existence and construction is given in the following Proposition:

Proposition 1. *Let $b < -1$, $0 < a \le 1$. The dynamics of map f in (3) can be investigated by using the first return map $F_r(x)$ in the interval $I = [0, 1]$. $F_r(x)$ is a discontinuous map with infinitely many branches defined as follows:*

$$F_r(x) := \begin{cases} F_{RL^{\overline{n}}}(x) = f_L^{\overline{n}} \circ f_R(x) & \text{if } \xi_{\overline{n}+1} \le x \le 1 \\ F_{RL^{\overline{n}+1}}(x) = f_L^{\overline{n}+1} \circ f_R(x) & \text{if } \xi_{\overline{n}+2} \le x < \xi_{\overline{n}+1} \\ \vdots & \vdots \\ F_{RL^{\overline{n}+j}}(x) = f_L^{\overline{n}+j} \circ f_R(x) & \text{if } \xi_{\overline{n}+j+1} \le x < \xi_{\overline{n}+j} \\ \vdots & \vdots \end{cases} \tag{4}$$

where $\overline{n} \geq 0$ is the smallest integer for which $f_L^{\overline{n}} \circ f_R(1) \in [0, 1)$,

$$F_{RL^m}(x) = \frac{a^m b}{x^\gamma} + \frac{1 - a^{m+1}}{1 - a} \tag{5}$$

and the discontinuity points are preimages of the origin given by

$$\xi_{m+1} = f_R^{-1} \circ f_L^{-m}(0) = \left(\frac{-b}{\frac{a^m - 1}{a^m(a-1)} + 1} \right)^{\frac{1}{\gamma}}.$$

Proof. Since in the region $b < -1$ of the parameter space $f_R(1) = 1 + b < 0$ holds, it is possible to define the first return map of $f(x)$ in the interval $[0, 1]$. We recall that the first return map $F_r(x)$ is defined as the function which associates to any point $x > 0$ the first non negative value of the trajectory of x, that is, the first value satisfying $f^n(x) \geq 0$, which in our case is necessarily $f^n(x) \in [0, 1)$. We also notice that when a point ξ satisfies $f^n(\xi) = 0$, then it is also $f_L \circ f^n(\xi) = 1$. So, given a value $b < -1$, let $\overline{n} \geq 0$ be the smallest integer such that

$$f_L^{\overline{n}} \circ f_R(1) \in [0, 1) \tag{6}$$

In the generic case satisfying $f_L^{\overline{n}} \circ f_R(1) \in (0, 1)$ we have that decreasing x from 1, $f_R(x)$ decreases and the first return map decreases as well, so that it must be defined as $F_r(x) = f_L^{\overline{n}} \circ f_R(x)$ for all the points in a left interval of 1, up to a point $\xi_{\overline{n}+1}$ in which it holds

$$f_L^{\overline{n}} \circ f_R(\xi_{\overline{n}+1}) = 0. \tag{7}$$

That is, $\xi_{\overline{n}+1}$ is a preimage of the origin of order $(\overline{n} + 1)$ as, taking the inverses in (7), we have $\xi_{\overline{n}+1} = f_R^{-1} \circ f_L^{-\overline{n}}(0)$. By applying f_L on both sides in (7), we also have that

$$f_L^{\overline{n}+1} \circ f_R(\xi_{\overline{n}+1}) = 1 \tag{8}$$

It follows that in a left neighborhood of the point $\xi_{\overline{n}+1}$ the first return map must be defined as $F_r(x) = f_L^{\overline{n}+1} \circ f_R(x)$, up to a point $\xi_{\overline{n}+2}$ in which it holds $f_L^{\overline{n}+1} \circ f_R(\xi_{\overline{n}+2}) = 0$, and so on. We can state that, for any $j \geq 0$, the first return map is defined by branches of the following kind:

$$F_{RL^{\overline{n}+j}}(x) = f_L^{\overline{n}+j} \circ f_R(x)$$

separated by discontinuity points (preimages of the origin).

The number of branches is necessarily infinite. In fact, as described above, we have to consider the preimages of the origin as follows:

$$\xi_{\overline{n}+j+1} = f_R^{-1} \circ f_L^{-(\overline{n}+j)}(0) \tag{9}$$

which exist for any $j \geq 0$. Considering the inverse functions

$$f_R^{-1}(y) = \left(\frac{b}{y - 1} \right)^{\frac{1}{\gamma}} \tag{10}$$

$$f_L^{-1}(y) = \frac{y - 1}{a} \tag{11}$$

the iterative application of the inverse on the left side leads to

$$f_L^{-k}(y) = \frac{y}{a^k} - \frac{a^k - 1}{a^k(a - 1)} \tag{12}$$

so that from (9), by using (12) and (10), we have explicitly:

$$\xi_{\overline{n}+j+1} = \left(\frac{-b}{\frac{a^{\overline{n}+j} - 1}{a^{\overline{n}+j}(a-1)} + 1} \right)^{\frac{1}{\gamma}} \tag{13}$$

We know that the points $f_L^{-(\overline{n}+j)}(0)$ exist on the left side for any $j \geq 0$, because the function f_L is increasing with slope $a \leq 1$, so that as $j \to \infty$ the points $f_L^{-(\overline{n}+j)}(0)$ tend to $-\infty$ and thus $f_R^{-1} \circ f_L^{-(\overline{n}+j)}(0)$ exist for any $j \geq 0$ and tend to 0.

The first return map is necessarily defined by infinitely many branches separated by discontinuity points, preimages of the origin of rank $(\overline{n}+j)$, denoted by $\xi_{\overline{n}+j}$. $F_r(x) = F_{RL^{\overline{n}}}(x) = f_L^{\overline{n}} \circ f_R(x)$ for $\xi_{\overline{n}+1} \leq x \leq 1$, and $F_r(\xi_{\overline{n}+1}) = 0$; by $F_r(x) = F_{RL^{\overline{n}+1}}(x) = f_L^{\overline{n}+1} \circ f_R(x)$ for $\xi_{\overline{n}+2} \leq x < \xi_{\overline{n}+1}$ which is a continuous increasing branch from 0 to 1, as $F_r(\xi_{\overline{n}+2}) = f_L^{\overline{n}+1} \circ f_R(\xi_{\overline{n}+2}) = 0$ and $F_r(\xi_{\overline{n}+1}) = f_L^{\overline{n}+1} \circ f_R(\xi_{\overline{n}+1}) = 1$, and so on, this holds for any integer. That is, for any j, $F_r(x) = F_{RL^{\overline{n}+j}}(x) = f_L^{\overline{n}+j} \circ f_R(x)$ is a continuous increasing branch for $\xi_{\overline{n}+j+1} \leq x < \xi_{\overline{n}+j}$, taking values from 0 to 1, as $F_r(\xi_{\overline{n}+j+1}) = f_L^{\overline{n}+j} \circ f_R(\xi_{\overline{n}+j+1}) = 0$ and $F_r(\xi_{\overline{n}+j}) = f_L^{\overline{n}+j} \circ f_R(\xi_{\overline{n}+j}) = 1$.

Examples of map f and its first return map F_r are shown in Fig.1.

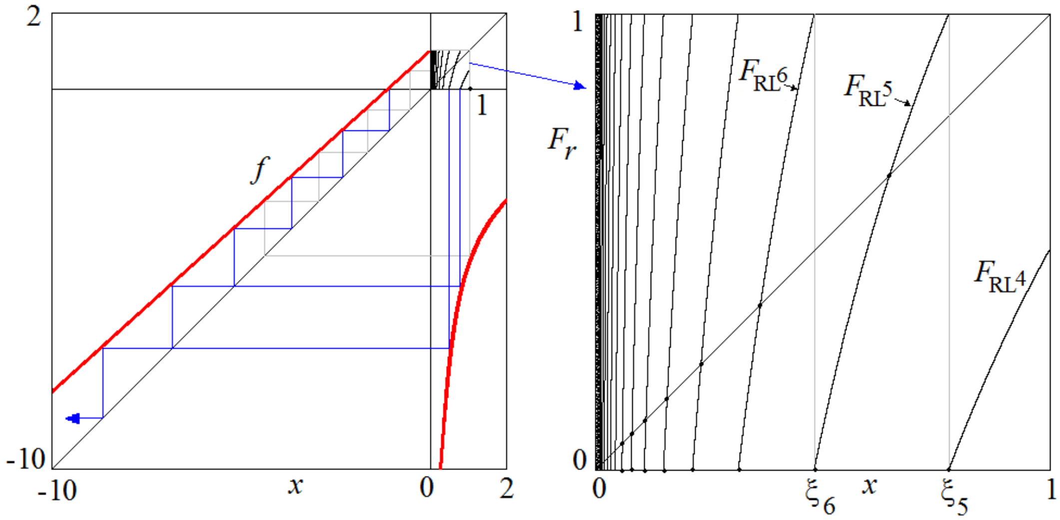

Figure 1: Map f at $\gamma = 0.5$, $a = 0.9$, $b = -5.5$, for which $\overline{n} = 4$ in the definition of $F_r(x)$ The function f is drown in red, the images of the point $x = 1$ are marked in gray, while a few preimages of $x = 0$ in blue. In the enlargement the related first return map $F_r(x)$.

In the particular case in which the condition in (6) occurs as $f_L^{\overline{n}} \circ f_R(1) = 0$, we also have $F_{RL^{\overline{n}+1}}(x) = f_L^{\overline{n}+1} \circ f_R(1) = 1$, so that we can define $F_r(1) = f_L^{\overline{n}} \circ f_R(1) = 0$ in the single point $\xi_{\overline{n}+1} = 1$ and then $F_r(x) = f_L^{\overline{n}+1} \circ f_R(x)$ in $[\xi_{\overline{n}+2}, \xi_{\overline{n}+1})$. Notice that the range of $F_{RL^{\overline{n}+1}}(x) = f_L^{\overline{n}+1} \circ f_R(x)$ in $[\xi_{\overline{n}+2}, 1]$ is exactly $[0, 1]$, and similarly, in all the other branches of $F_r(x)$ which can be defined as above. Examples are shown in Fig.2.

We have so proved that the first return map in $[0, 1]$ is a discontinuous map defined by infinitely many increasing branches as explicitly given in (4).□

2.1 Border collision bifurcations

The particular case

$$f_L^{\overline{n}} \circ f_R(1) = 0 \tag{14}$$

mentioned in the proof given above can be rewritten as $f_L^{\overline{n}} \circ f_R \circ f_L(0) = 0$ (considering $1 = f_L(0)$), or equivalently, by applying f_L on both sides of (14), as follows:

$$F_{RL^{\overline{n}+1}}(1) = f_L^{\overline{n}+1} \circ f_R(1) = 1 \tag{15}$$

thus it corresponds to the BCB of a basic cycle with symbolic sequence $RL^{\overline{n}+1}$ (as in fact $x = 0$, as well as $x = 1$, is a periodic point of period $(\overline{n}+2)$).

In terms of the preimages of the origin the condition in (14) also corresponds to

$$1 = f_R^{-1} \circ f_L^{-\overline{n}}(0) \tag{16}$$

that is, by using the definition in (9) with $j = 0$,

$$\xi_{\overline{n}+1} = 1 \tag{17}$$

The equation $\xi_{\overline{n}+1} = 1$ of the BCB can be written in explicit form. In fact, considering $n = \overline{n} + 1$ in (13) and $j = 0$ we have

$$1 = \left(\frac{-b}{\frac{a^n - 1}{a^n(a-1)} + 1} \right)^{\frac{1}{\gamma}}$$

equivalent to

$$-b = \frac{a^n - 1}{a^n(a-1)} + 1$$

and, rearranging:

$$B_{RL^n} : \; b = -\frac{1 - a^n}{a^{n-1}(1-a)} \tag{18}$$

which is the equation of the BCB of a cycle with symbolic sequence RL^n.

Considering the example shown in Fig.1, as b increases from the value -5.5, the value $F_{RL^4}(1) = f_L^4 \circ f_R(1)$ of the rightmost branch of F_r increases, and when $F_{RL^4}(1) = f_L^4 \circ f_R(1) = 1$, from (15) the BCB of the cycle with symbolic sequence RL^4 occurs, as shown in Fig.2a (from (18) with $a = 0.9$ and $n = 4$ the bifurcation value $b \simeq -4.71742$ is obtained). Differently, as b decreases from the value -5.5, the value $F_{RL^4}(1) = f_L^4 \circ f_R(1)$ of the rightmost branch of F_r decreases, and when $F_{RL^4}(1) = f_L^4 \circ f_R(1) = 0$ (which corresponds to $f_L^5 \circ f_R(1) = 1$) from (14) the BCB of the cycle with symbolic sequence RL^5 occurs, as shown in Fig.2b (from (18) with $a = 0.9$ and $n = 5$ the bifurcation value $b \simeq -6.24158$ is obtained). As we shall see in Sec.4, as b decreases up to $-\infty$, all the BCB curves of cycles with symbolic sequence RL^k for $k > 4$ are crossed, and the first return map is expansive (the first derivative of all the component branches $F_{RL^m}(x)$ is larger than 1 in all the points of the related intervals).

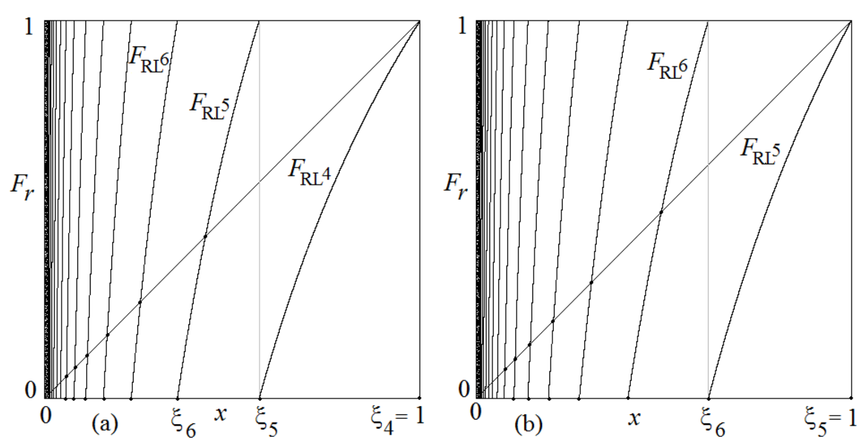

Figure 2: First return map $F_r(x)$ at $\gamma = 0.5$, $a = 0.9$. In (a) $b = -4.71742$ BCB value of the maximal cycle RL^4. In (b) $b = -6.24158$, BCB value of the maximal cycle RL^5.

A few properties of the first return map F_r immediately follow.

(i) Each component $F_{RL^n}(x)$ of F_r is continuous and increasing from 0 to 1, except at most the rightmost branch (as in the example in Fig.1), as $F'_{RL^n}(x) > 0$, for $x > 0$. This also follows from the explicit expression of the first derivative:

$$F'_{RL^n}(x) = a^n f'_R(x) = \frac{-b\gamma}{x^{\gamma+1}} a^n > 0$$

(ii) Each component $F_{RL^n}(x)$ of F_r is concave, since $F''_{RL^n}(x) < 0$, for $x > 0$, as follows from the explicit expression:

$$F''_{RL^n}(x) = \frac{d}{dx}(\frac{-b\gamma}{x^{\gamma+1}}a^n) = \frac{b\gamma(\gamma+1)}{x^{\gamma+1}}a^n < 0$$

(iii) The same properties (increasing branches and concavity) hold for any composition of the functions $F_{RL^n}(x)$.

(iv) An immediate consequence of the constructive definition of the first return map F_r, is that infinitely many unstable basic cycles necessarily exist. In fact, the first return map $F_r(x)$ consists of infinitely many increasing and concave branches $F_{RL^j}(x)$ which are continuous and take values from 0 to 1, at least for any $j \geq \overline{n}+1$. Thus unstable fixed points x_{RL^j} must exist for any $j \geq \overline{n}+1$.

In the example given in Fig.1, where $\overline{n} = 4$, the rightmost branch of F_r is defined by $F_{RL^4}(x) = f_L^4 \circ f_R(x)$. All the branches defined by $F_{RL^{4+j}} = f_L^{4+j} \circ f_R(x)$ exist for any $j \geq 1$ and intersect the diagonal, leading to the existence of unstable cycles of period $(5+j)$ for any $j \geq 1$.

3 Chaos in a Baker-like map with infinitely many discontinuity points

In the previous section we have seen several properties of the map f that can be studied by the first return map $F_r : I \to I$ having three peculiarities: infinitely many discontinuity points which have $x = 0$ as limit point, all the continuous branches $F_{RL^j}(x)$ of F_r take values from 0 to 1 except at most the rightmost one and, as we shall prove in the next section, all the functions of the component branches are expanding. In this section we prove that a map having these properties, which we call Baker-like map with infinitely many discontinuity points or equivalently with infinitely many branches, is chaotic in I. Clearly, for the map f this means that the whole *unbounded interval* $(-\infty, 1]$ is a chaotic attractor (whose robustness will be proved in Sec.4).

Definition 1 (Baker-like). *A function $\phi : [0,1] \longrightarrow [0,1]$ defined by*

$$\phi(x) = \begin{cases} \phi_1(x) & \text{if } \xi_1 \leq x \leq \xi_0 \\ \phi_2(x) & \text{if } \xi_2 \leq x < \xi_1 \\ \vdots & \vdots \\ \phi_i(x) & \text{if } \xi_i \leq x < \xi_{i-1} \\ \vdots & \vdots \\ 0 & \text{if } x = 0 \end{cases}$$

is called a 1D Baker-like map with infinitely many discontinuouty points in $I = [0,1]$ if the $\{\xi_i\}_{i=0}^{\infty} \subset I$ constitute a decreasing sequence of positive numbers with $\xi_0 = 1$ such that $\lim_{i\to\infty} \xi_i = 0$ and ϕ_i a family of differentiable functions

$$\phi_i : [\xi_i, \xi_{i-1}] \longrightarrow [0,1]\,, \quad \text{for any } i \geq 1$$

satisfying $\phi_i(\xi_i) = 0$ for any $i \geq 1$, $\phi_i(\xi_{i-1}) = 1$ for any $i > 1$, $0 \leq \phi_1(1) \leq 1$, and $\phi'_i(x) > 1$ for any $x \in [\xi_i, \xi_{i-1}]$.

In the following lemma, we prove that any open interval in I has an image of finite rank which includes at least one discontinuity point.

Lemma 1. *Let ϕ be a 1D Baker-like map with infinitely many discontinuity points in $I = [0,1]$. Then for any interval $J = (\alpha, \beta) \subset I$, there is $k \geq 0$ such that*

$$\phi^k(J) \cap \{\xi_i \mid i \in \mathbb{N}\} \neq \varnothing.$$

Proof. Reasoning by contradiction, suppose that there is an interval $J = (\alpha, \beta) \subset I$ such that

$$\phi^k(J) \cap \{\xi_i \mid i \in \mathbb{N}\} = \varnothing$$

for any $k \geq 0$. For $k=0$, J does not include any discontinuity point, so there is $i_0 \geq 1$ such that $J \subset (\xi_{i_0}, \xi_{i_0-1})$ and $\phi(J)$ is an interval. Since ϕ is continuous and increasing on J, by the *mean value theorem* there is $c_1 \in J$ such that

$$|\phi(J)| = |\phi(\beta) - \phi(\alpha)| = \mu_1(\beta - \alpha) = \mu_1 |J|$$

where $\phi'(c_1) = \mu_1 > 1$. Now consider $k=1$, then there exists $i_1 \geq 1$, such that $\phi(J) \subset (\xi_{i_1}, \xi_{i_1-1})$ and $\phi^2(J)$ is an interval. Similarly there is $c_2 \in \phi(J)$ such that

$$\left|\phi^2(J)\right| = \mu_2 |\phi(J)| = \mu_1\mu_2 |J|$$

where $\left(\phi^2(c_2)\right)' = \mu_2 > 1$. So iteratively, for $k=n$ there is $i_n \geq 1$ such that $\phi^n(J) \subset (\xi_{i_n}, \xi_{i_n-1})$ and $\phi^{n+1}(J)$ is an interval. Likewise, there is $c_n \in \phi^n(J)$ satisfying

$$\left|\phi^{n+1}(J)\right| = \mu_n |\phi^n(J)| = (\mu_1\mu_2\cdots\mu_n) |J|$$

where $(\phi^{n+1}(c_n))' = \mu_n > 1$. Since $\mu_i > 1$, we have

$$\lim_{n\to+\infty} \left|\phi^{n+1}(J)\right| = +\infty$$

which is a contradiction.□

We prove that a 1D Baker-like map with infinitely many discontinuity points is chaotic in the sense of Devaney [7] in $I=[0,1]$. Let us recall the following

Definition 2 (chaos). *Let (X,d) be a metric space without isolated points. Then a dynamical system $\phi : X \longrightarrow X$ is said to be chaotic (in the sense of Devaney) if it satisfies the following conditions:*

(1) **transitivity:** *ϕ is topologically transitive in X; that is, for any pair of non-empty open sets U and V of X there exists a natural number n such that $\phi^n(U) \cap V \neq \varnothing$;*

(2) **density:** *the periodic points of ϕ are dense in X;*

(3) **sensitivity:** *ϕ has sensitive dependence on initial conditions in X; that is, there is a positive constant δ (sensitivity constant) such that for every point x of X and every neighborhood N of x there exists a point y in N and a non negative integer n such that $d(\phi^n(x), \phi^n(y)) \geq \delta$.*

If ϕ is continuous, one can drop the sensitivity condition from Devaney's definition of chaos because it is implied by the other two conditions ([4]). Moreover, it has been proved that if ϕ is a continuous map on an interval, not neccesserily a finite interval, then transitivity implies density and sensitivity ([25]). Namely, for continuous maps on an interval, both sensitivity and density are redundant conditions in the definition of chaos. However, for the 1D discontinuous maps which we are interested in, the three conditions have to be proved separately.

Theorem 1. *Let ϕ be a 1D Baker-like map with infinitely many discontinuouty points in $I=[0,1]$. Then, ϕ is chaotic in $I=[0,1]$.*

Proof. We show that ϕ satisfies the conditions in Definition 2 (transitivity, density and sensitivity).

(1) First we prove transitivity. Let $J=(\alpha,\beta) \subset I$ be an arbitrary open interval. According to Lemma 1, let k_0 be the smallest positive integer such that $\phi^{k_0}(J) \cap \{\xi_i \mid i \in \mathbb{N}\} \neq \varnothing$. So, there is at least one discontinuity point, say $\xi_j \in \phi^{k_0}(J) \cap \{\xi_i \mid i \in \mathbb{N}\}$. We know that ϕ_j is continuous and increasing on $[\xi_j, \xi_{j-1})$ and $\phi_j(\xi_j) = 0$. Since $\xi_j \in \phi^{k_0}(J)$, we have that $0 = \phi(\xi_j) \in \phi^{k_0+1}(J)$. Since 0 is limit set of the discontinuity points, there exists n_0 such that $\xi_n \in \phi^{k_0+1}(J)$ for any $n \geq n_0$. Hence $[\xi_n, \xi_{n-1}] \subset \phi^{k_0+1}(J)$. Now by applying ϕ on both sides, we obtain $[0,1) = \phi([\xi_n, \xi_{n-1}]) \subset \phi^{k_0+2}(J)$. So it is topologically mixing, and therefore topologically transitive.

(2) Regarding density, let $J=(\alpha,\beta) \subset I$ be an arbitrary interval. By Lemma 1, let k_0 be the smallest positive integer such that $\phi^{k_0}(J) \cap \{\xi_i \mid i \in \mathbb{N}\} \neq \varnothing$. We know that $\phi^{k_0}(J) = \left(\phi^{k_0}(\alpha), \phi^{k_0}(\beta)\right)$ is an interval. Let ξ_j be the largest discontinuouty point in $\phi^{k_0}(J)$. Note that ϕ^{k_0} is continuous and increasing on J and $\xi_j \in \phi^{k_0}(J)$. So there exists $x_0 \in J$, such that $\phi^{k_0}(x_0) = \xi_j$ and $[x_0, \beta) \subset J$. Since $\phi^{k_0+1}(x_0) = \phi(\phi^{k_0}(x_0)) = \phi(\xi_j) = 0$ and 0 is limit set of the discontinuity points, there is n_0 such that $\xi_n \in \phi^{k_0+1}([x_0, \beta))$ for any $n \geq n_0$. It follows that $[\xi_n, \xi_{n-1}] \subset \phi^{k_0+1}([x_0, \beta))$. Thus there are $x_1 \in [x_0, \beta)$ such that $\phi^{k_0+1}(x_1) = \xi_n$ and also $y_1 \in [x_0, \beta)$ satisfying $\phi^{k_0+1}(y_1) = \xi_{n-1}$. Clearly, $[x_1, y_1] \subset [x_0, \beta)$ and $\phi^{k_0+2}([x_1, y_1)) = [0,1)$.

Since $[x_1, y_1) \subset [0,1)$, let $c = \frac{y_1+1}{2}$. It is clear that $y_1 < c < 1$. Moreover, there is $d \in [x_1, y_1)$ such that $\phi^{k_0+2}(d) = c$. Now we define a new map

$$g : [x_1, d] \longrightarrow [0, c]$$

such that $g(x) := \phi^{k_0+2}(x) - x$. Since g is continuous on $[x_1, d]$ and $g(x_1) = 0 - x_1 = -x_1 < 0$ and $g(d) = c - d > 0$, by *Bolzano' theorem* there exists $x_* \in [x_1, d] \subset J$ such that $g(x_*) = \phi^{k_0+2}(x_*) - x_* = 0$, that is $\phi^{k_0+2}(x_*) = x_*$. This completes the proof.

(3) For the proof of sensitivity, we show that there exists $\delta > 0$ such that for any $p \in I$ and any neighborhood U *of* p, there is $q \in U$ and $j \geq 1$ such that $d(\phi^j(p), \phi^j(q)) \geq \delta$. Fix $\delta = \frac{1}{2}$. According to the proof of part (i), there is $j \geq 1$ such that $\phi^j(U) = [0,1)$. Let $\phi^j(p) = p_j \in I$ and $q_j \in I$ such that $|q_j - p_j| = \frac{1}{2}$. Since $q_j \in I$, there is $q \in U$ such that $\phi^j(q) = q_j$ and thus

$$d(\phi^j(p), \phi^j(q)) \geq \delta.$$

□

4 Robust unbounded chaotic attractors

In this section we prove that in the considered example of map f given in (3) there are open sets in the parameter space at which the dynamics of the system persist as chaotic in the unbounded interval $(-\infty, 1]$, showing that the first return map F_r defined in Sec.2 is a Baker-like map with infinitely many discontinuity points.

To this purpose, let us recall some features of the bifurcation curves in the parameter space of our map. Besides the BCB B_{RL^n} determined in Sec.2 associated with cycles having symbolic sequence RL^n ($n \geq 2$, in the considered range $b < -1$) it is known that fold bifurcations of basic cycles may occur (see [20], [14]). A fold bifurcation leads to two merging solutions of the equation $F_{RL^n}(x) = x$, and from (5) this leads to

$$\frac{a^n b}{x^\gamma} + \frac{1 - a^{n+1}}{1 - a} = x. \tag{19}$$

The eigenvalue of a cycle is the first derivative of the composite function $F_{RL^n}(x)$, thus we have

$$F'_{RL^n}(x_0) = \frac{-b\gamma}{x_0^{\gamma+1}} a^n \tag{20}$$

where x_0 is the periodic point on the R side. Taking into account that at a fold bifurcation two fixed points are merging in one point denoted $x^*_{RL^n}$ and that $F'_{RL^n}(x^*_{RL^n}) = 1$, from (20) we obtain the condition

$$x^*_{RL^n} = (-b\gamma a^n)^{\frac{1}{\gamma+1}} \tag{21}$$

By substituting this expression into (19), the equation of the fold bifurcation in the function F_{RL^n} (a curve in the parameter plane (a, b)) is obtained, for any $n \geq 1$, given by:

$$\Phi_{RL^n} : \; b = -\frac{1}{\gamma a^n} \left(\frac{1 - a^{n+1}}{1 - a} \frac{\gamma}{\gamma + 1} \right)^{\gamma+1} \tag{22}$$

It is worth to note that for any $n \geq 1$ (and any $\gamma > 0$, $a > 0$) the two curves Φ_{RL^n} and B_{RL^n} have a point of tangency, as can be observed in Fig.3 where a few BCB curves B_{RL^n} are shown in black and the fold bifurcation curves Φ_{RL^n} in red, and the points of tangency are marked by black circles. Each codimension-two point, say $(\overline{a}_n, \overline{b}_n)$, satisfies both equations in (22) and (18), thus $\overline{a}_n$ can be obtained as the unique solution of the following equation:

$$a\gamma \frac{1 - a^n}{1 - a} - \left(\frac{1 - a^{n+1}}{1 - a} \frac{\gamma}{1 + \gamma} \right)^{\gamma+1} = 0 \tag{23}$$

and $\overline{b}_n = b(\overline{a}_n)$ is obtained from (18) at $a = \overline{a}_n$. The equation in (23) cannot be easily solved analytically. However, a simpler expression can be obtained. In fact, considering that at the codimension-two point both bifurcations must occur simultaneously, and since the border collision occurs when the periodic point on the right side collides with $x = 1$, we can state that the fold bifurcation point given in (21), $x^*_{RL^n} = (-b\gamma a^n)^{\frac{1}{\gamma+1}}$, must be equal to 1, that is, simplifying:

$$-b\gamma a^n = 1 \tag{24}$$

By substituting $b = -\frac{1}{a^n\gamma}$ into (18) we get $-\frac{1}{a^n\gamma} = -\frac{1-a^n}{a^{n-1}(1-a)}$, that is $a\frac{1-a^n}{1-a} = \frac{1}{\gamma}$ and, rearranging:

$$a^{n+1} - a\left(1 + \frac{1}{\gamma}\right) + \frac{1}{\gamma} = 0 \tag{25}$$

For the BCB curve of the 2−cycle RL ($n = 1$), occurring at $b = -1$, the condition in (24) reduces to $a - \frac{1}{\gamma} = 0$, so that the codimension-two point is given by

$$\overline{a}_1 = \frac{1}{\gamma} \tag{26}$$

While for $n = 2$, related to the BCB curve of the 3−cycle RL^2, we obtain

$$\overline{a}_2 = \frac{1}{2}\left(-1 + \sqrt{1 + \frac{4}{\gamma}}\right).$$

In addition, it follows that increasing n the solutions are decreasing values (i.e. $\overline{a}_{n+1} < \overline{a}_n$), and all these values are larger than a fixed value of a which can be obtained from (25) as $n \to \infty$, leading to

$$a_\infty = \frac{1}{\gamma + 1}. \tag{27}$$

So we can state that, for any $n > 1$, the following inequalities hold:

$$a_\infty = \frac{1}{\gamma + 1} < \overline{a}_{n+1} < \overline{a}_n < \overline{a}_1 = \frac{1}{\gamma}. \tag{28}$$

When the parameters belong to the BCB curve B_{RL^n} then a periodic point is merging with $x = 1$, it holds $F'_{RL^n}(1) = -b\gamma a^n = \gamma a^n \frac{1-a^n}{a^{n-1}(1-a)} = \gamma a \frac{1-a^n}{1-a}$ and

- for $a < \overline{a}_n$ we have $F'_{RL^n}(1) < 1$ which means that the colliding cycle is stable, and thus the fold bifurcation curve Φ_{RL^n} (associated with a point in which $F'_{RL^n} = 1$) must have been occurred before at a smaller value of b;
- for $a > \overline{a}_n$ we have $F'_{RL^n}(1) > 1$ which means that the colliding cycle is unstable, and thus the fold bifurcation curve (associated with a point in which $F'_{RL^n} = 1$) must be virtual (at larger values of a), below the BCB curve B_{RL^n}.

The codimension-two points on a BCB curve separate different dynamic behaviors. If we consider a point of a BCB curve B_{RL^n} at the right of its codimension-two point $(\overline{a}_n, \overline{b}_n)$ it holds that $F'_{RL^n}(1) > 1$ and the colliding cycle is unstable.

The case associated with the example in Fig.2a, related to the curve B_{RL^4}, corresponds to the upper point in the blue segment marked by an arrow in Fig.3a, while the example in Fig.2b, related to the curve B_{RL^5}, corresponds to the lower point. The case shown in the enlargement of Fig.1 corresponds to a point inside the segment.

For parameters $(\overline{a}, \overline{b}) \in B_{RL^n}$ and $\overline{a} \geq \overline{a}_n$, the first return map consists of infinitely many branches $F_{RL^j}(x)$, $j \geq n$, and all of them, including the rightmost one $F_{RL^n}(x)$, have range $[0, 1]$. Since $F'_{RL^n}(1) > 1$, then it must be $F'_{RL^n}(x) > 1$ for any $x \in [\xi_{n+1}, 1)$. Notice that the codimension-two points $\overline{a}_j$ of $B_{RL^j}(x)$, $j > n$, all are smaller than $\overline{a}_n$ which means that at fixed $\overline{a}$ decreasing b all the BCB curves $B_{RL^j}(x)$, $j > n$ are crossed and at such bifurcation points it holds $F'_{RL^j}(1) > 1$ for any $j > n$. This implies that

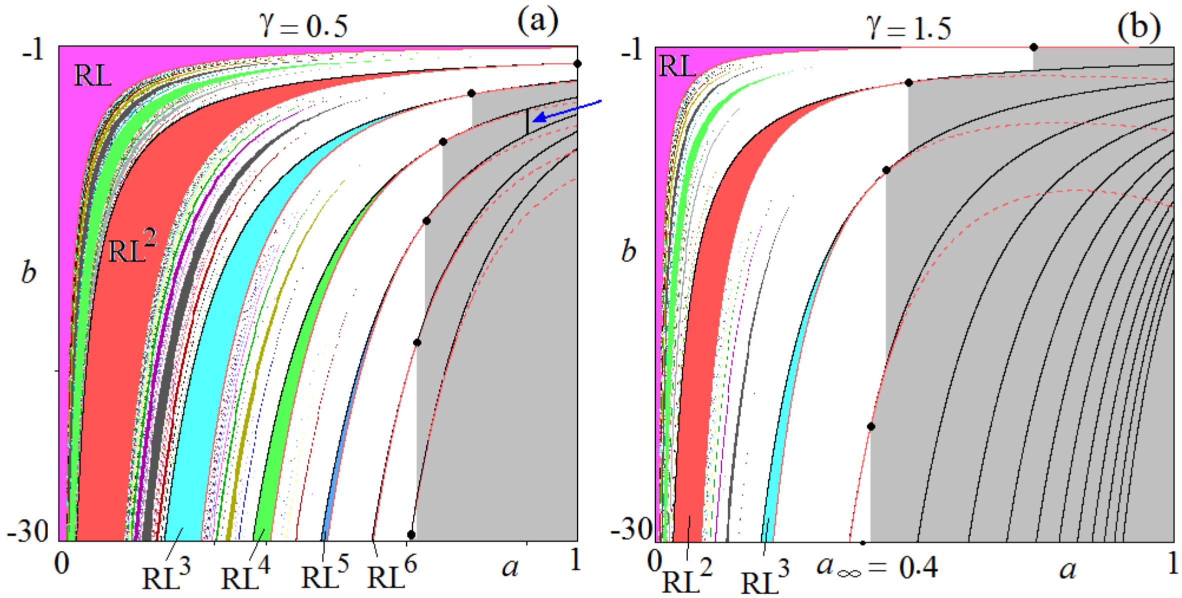

Figure 3: Two-dimensional bifurcation diagram in the (a, b) parameter plane. In (a) at $\gamma = 0.5$, in (b) at $\gamma = 1.5$. Regions related to stable cycles of different periods are shown in color. The periodicity regions of the maximal cycles RL^n are evidenced. The lower boundary (in red) is a fold bifurcation curve Φ_{RL^n} while the upper boundary (in black) is a BCB curve B_{RL^n}. The bifurcation curves Φ_{RL^n} and B_{RL^n} are drawn by using the equations given in (22) and (18), respectively. The codimension two points $(\overline{a}_n, \overline{b}_n)$ are marked with black circles. In (a) $a_\infty = \frac{2}{3}$, in (b) $a_\infty = \frac{2}{5}$. The segment in (a) evidenced by the blue arrow is at $a = 0.9$.

at $(\overline{a}, \overline{b}) \in B_{RL^n}$ also all the other branches, given by $F_{RL^j}(x)$, $j > n$, are expansive. In fact, the slope is certainly $F'_{RL^j}(x) > 1$ for $x \in [\xi_{j+1}, x^*_{j+1}]$ where x^*_{j+1} is the unstable fixed point of $F_r(x)$, then for $x \in [x^*_{j+1}, \xi_j]$ the slope, although decreasing, is larger than 1 as at the considered parameter ($\overline{a} > \overline{a}_n$) it cannot cross the value 1 (a branch $F_{RL^j}(x)$ of the first return map can have points with slope smaller than 1 only if at fixed value of a, decreasing b the fold bifurcation curve Φ_{RL^j} is crossed, which can occur for $a < \overline{a}_j$ and this cannot occur at the considered parameter).

This proves that the first return map is expanding, and thus $F_r(x)$ is a Baker-like map with infinitely many branches, at the points $(\overline{a}, \overline{b}) \in B_{RL^j}$ (where $\overline{a} \geq \overline{a}_n$) for any $j \geq n$. But the same result holds not only at the BCB values. In fact, considering any point $(\overline{a}, \overline{b}) \in B_{RL^n}$ with $\overline{a} \geq \overline{a}_n$, then *for any* $b \leq \overline{b}$ it is $F'_r(1) > 1$ and thus the rightmost branch of $F_r(x)$ has the slope larger than 1 in all its points (due to monotonicity and concavity), as in the example shown in the enlargement of Fig.1. Then, not only the rightmost branch, but also all the other (infinitely many) branches defining the first return map $F_r(x)$ have the slope larger than 1 in all the points. In fact, reasoning as above, the related branches all have an unstable fixed point, with slope larger than 1, and on its right side the slope, although decreasing, cannot cross the value 1 as this cannot occur for the considered parameter ($\overline{a} > \overline{a}_n$).

We have so proved that for any fixed $\gamma > 0$ considering a BCB curve B_{RL^n}, in all the points (a, b) of the two-dimensional bifurcation diagram with $a \geq \overline{a}_n$ and $b \leq \overline{b}_n$ the first return map $F_r(x)$ is a Baker-like map with infinitely many branches (as in the gray region shown in Fig.3a,b). It follows that a wide area in the parameter space corresponds, for f, to the existence of a robust unbounded chaotic attractor, the interval $(-\infty, 1]$.

As we can see from Fig.3, the larger the value of γ, the wider is the region in the parameter space with robust unbounded chaotic attractors.

5 Conclusions

In this work we have proved the existence of robust full measure unbounded chaotic attractors in a discontinuous piecewise smooth one-dimensional map f, linear-hyperbolic, belonging to the family known as Nordmark's map. We have shown that the dynamics of the system in the considered parameter range can be studied by use of a suitable first return map, which has been analytically described. This first return map of the interval $I = [0, 1]$ into itself has infinitely many discontinuity points ξ_j which are accumulating to $x = 0$. In Sec.4 we have proved that in the considered parameter space all the branches are defined by expanding functions and have range I except at most the rightmost one, defined in a given interval $[\xi_{\overline{n}+1}, 1]$, whose range can be any interval $[0, y]$ with $0 < y \leq 1$. This kind of map has been called Baker-like map with infinitely many branches, and in Sec.3 we have proved that it is chaotic in I, proving that in I transitivity occurs, periodic points are dense and there is sensitivity with respect to the initial conditions. Proving that the first return map is chaotic in I we have proved that $(-\infty, 1]$ is an unbounded chaotic attractor of map f which is persistent under parameter perturbation in a set of positive measure of the parameter space.

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