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Optimal Waste Control with Abatement and Productive Capital Stocks*

Enrico Saltari† and Giuseppe Travaglini‡§

Abstract

In this paper we address the control problem of a social optimum in presence of waste and capital stocks. We address this problem in two stages. In the first, we suppose that output is fixed; next, we endogenize output allowing for growth. The analytical framework is simple. Consumption is assumed to generate an undesirable residue. Society can control waste accumulation using abatement capital, and rise output using productive capital which accumulates over time. We have three main results. (1) On the analytical ground we are able to find a closed form solution to the optimal consumption with waste, abatement and productive capital stocks. (2) For the case of fixed output, we get a solution where stocks and flows affect the dynamics of the system. Environmental policies may have permanent effects on the level of variables. Then, (3) when waste and abatement capital are embedded in a classical growth model, we obtain an Environmental Keynes-Ramsey rule which states that the growth rate of the productive capital is positive if and only if its net marginal productivity is greater than the net social cost it generates, given by the marginal disutility of waste weighted by its shadow cost.

Key words: Abatement capital; waste accumulation; optimal control; Pigouvian taxes and subsidies; output growth

JEL classification codes: E22, L51, H23, Q28.

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1 Introduction

In *Polluted Golden Age* Brock (1977) contended that economists generally tend to overlook the process of waste accumulation associated with economic activities. As waste accumulates over time, he remarked, it provides disutility to society. An array of environmental policies can be promoted to foil these detrimental phenomena. Investment in abatement capital can be done to control waste emissions, even though it takes time. Pigouvian taxes and subsidies help in controlling emissions, but different fiscal tools may have different impacts on emissions. Binding quotas and permits can limit waste externality, but they are difficult to implement because of the problem of commons.

In what follows, we will think over these issues. Our aim is twofold. First of all, we want to provide a *non* trivial closed form solution to the problem of welfare maximization with waste and abatement capital stocks. In the model presented here the two state variables, waste and abatement capital stocks, are strictly interrelated and the resulting intertemporal problem is quite complex. From a technical point of view, when there is more than a state variable the maximum principle continues to provide a set of necessary and sufficient conditions for an optimum, and if all functional forms and restrictions are fully specified it gives an explicit solution to the problem. We get a closed form solution both for the case of waste accumulation and abatement capital stock with fixed output (two state variables) and for the case of output growth where capital stock can also be used for production (three state variables). This is a step forward with respect to the previous literature on waste control where it is hard to find an explicit solution even for the first case of two state variables, namely waste and abatement capital stocks. This solution allows us – as an additional outcome – to study the timing and persistence of environmental policies aimed at controlling waste accumulation. Pigouvian taxes and subsidies are one of the objects of our analysis. From our model emerges that the trade-off between waste and abatement capital stock determines the relative desirability of taxes and subsidies in terms of social welfare and environmental quality.

As it is well known, the issue of waste control is not new in the economic-ecological debate. The first reference is Keeler, Spence and Zeckhauser (1971). They use a model where output is allocated to either consumption, investment or reduction of the emission-output ratio. The control variables are the shares of output devoted to consumption and clean technology. However, in that paper the stock of waste affects welfare, but abatement capital stock is not included in the model. Later, Plourde (1972), Forster (1973) Gruver (1976) and Lusky (1976) employ optimal control techniques to study the intertemporal effects of waste externality, but they consider only the effect of *flow* of pollution on welfare. An important advancement is in Smith (1972) which assumes that the welfare function is affected by pollution stock, and that waste is an inevitable by-product of production. It is worthwhile to note that Smith considers abatement activities as a flow (recycling). A similar approach characterizes the papers by Maler (1974) and Becker (1982) where pollution is a flow caused by consumption and production decisions, respectively. However, they do not allow for abatement capital stock. Only in Musu (1989) capital can be used for production and abatement. But, he does not take into account the direct effect of pollution on welfare.

Then, related to the issue of waste control is the matter of the relationship between waste accumulation and economic growth (Forster, 1975; Nordhaus, 1982, 1993; Weitzman, 1976; Solow, 1986; Hartwick, 1990; Van der Ploeg and Withagen, 1991; Withagen, 1995; Bovenberg and Smulders, 1995; Smulders and Gradus, 1996; Dasgupta and Maler, 2000; Brock and Starrett, 2003; Xepapadeas 2005). Among these it is worthwhile to mention Van der Ploeg and Withagen (1991). They focus on optimal emission charges in the context of the classical Ramsey problem. To get a closed form solution, they must restrict the number of state variables which impinge upon the dynamics of the economic-ecological system. Precisely, Van der Ploeg and Withagen consider two alternative scenarios. In the first one, they assume that society creates *activities* to clean up the environment. In the second scenario, society directly invests in a *clean technology.* Both these abatement activities are treated, however, as flows which contrast the emission of pollutants period by period. Therefore, their analysis distinguishes between stock and flow externalities arising from waste, but does not allow to investigate the long lasting effect of abatement capital stock on waste accumulation. As a consequence, the study of the relationship between abatement capital and waste stock, and its effect on welfare and economic growth, still remains an open question.

Finally, in order to complete our survey it is necessary to take into account the models based on the cost-benefit analysis. In its most updated formulation, the cost-benefit analysis is concerned with the optimal timing of a discrete policy that a society should adopt to reduce waste emissions. Uncertainty and irreversibility are at the core of this class of models, and real option is the more appropriate technique to formalize this type of problem (Smith, 1987; Pindyck, 2000, 2002; Xepapadeas 2001; Lin et al*.*, 2007; Ansar and Spark, 2009; Agliardi and Sereno, 2012). An interesting application is in Xepapadeas (1992), Lin and Huang (2010, 2011) and Saltari and Travaglini (2011a, 2011b). In all these papers the attention shifts from the social welfare to the private net benefits gained from a green firm which invests in abatement capital. Specifically, the "private view" explains why environmental policies, promoted to stimulate clean activities, can sometimes generate unexpected allocation of resource, reducing, instead of increasing, abatement capital stock over time.

The present paper improves on the previous literature in several dimensions. Basically, we consider a problem of waste control where abatement capital stock influences the accumulation of waste. Welfare is negatively affected by waste, while it is positively affected by consumption. On the other hand, consumption generates the undesirable residue which contributes to increase waste stock. Therefore, in our set up abatement capital indirectly influences welfare over time. The intertemporal process feeds the economic-ecological trade-off between waste accumulation and abatement capital, and affects both the dynamics and the steady state of the economic-ecological system.

In our opinion, this issue is of crucial importance for sustainable economic growth. An example of this problem is to be found in the recycling of plastic and steel cans. The optimal allocation of waste along time depends not only on the relationship between private and social costs, but also on the ability of abatement capital to dispose of polluting emissions caused by consumption and to control waste accumulation over time. One can speculate that even the recent dramatic facts of the steelworks ILVA of Taranto in the South of Italy exemplifies this problem, and makes tangible the implicit cost inherent in the economic-ecological trade-off, and the impact of this cost in term of welfare and environment.

The added value of the paper lies in providing a closed form solution to the environmental problem of welfare maximization with both waste, abatement and productive capital stocks. Indeed, the structure of our problem is such that we can give a non trivial solution to two progressively more complex forms of this maximization problem. We first study the benchmark case with waste and abatement capital stocks assuming output fixed. We focus on the consequences of environmental policies aimed at controlling waste accumulation. Indeed, having a closed form solution makes it possible to analyze the timing and impacts of taxes and subsidies on waste emissions and abatement investments. Next, we study the waste control problem if we allow capital to be used also in production and thus output to grow.

We get two main results.

Firstly, for a society the problems of abatement capital decisions, waste and consumption control are strictly interrelated. Any change in abatement capital stock affects waste accumulation and consumption. Without output growth, the waste charge corresponds to the social cost of an additional unit of waste. We show that the adoption of either temporary subsidies to stimulate investment, or temporary taxes to discipline consumers do not change the long-run equilibrium of the economy. Indeed, it is not just current subsidies or taxes, but their entire path over time that affects investment decisions and waste accumulation. Thus, expectations about how long the environmental policies would last have a special role in affecting investment decisions and waste accumulation.

Secondly, when waste and abatement capital are embedded in a Ramsey model, we get an Environmental Keynes-Ramsey rule where the growth rate of capital accumulation is positive if and only if the net marginal product of capital is greater than the net social cost it generates, given by the marginal disutility of waste weighted by its shadow cost. This condition is a special version the famous Keynes-Ramsey rule for consumption.

The organization of the paper is as follows. In the next section we present the model with exogenous output. In section 3 we study the dynamic properties of the model using phase diagrams. The effects of permanent and temporary taxes and subsidies are analyzed in section 4. From section 5 to 7 we endogenize output and study the problem of pollution control in a Ramsey model with additional costs associated with productive capital. Section 8 concludes the paper.

2 Waste control: the benchmark case

Following Plourde (1972), Smith (1972), Forster (1973, 1975) and Withagen (1995), consider a society consuming a good and which, as a by-product of this consumption, increases the stock of waste. The social utility function at time t is

$$u(C_t, M_t) \tag{1}$$

where C_t is consumption and M_t the stock of waste. We assume that consumption is a good ($\frac{\partial u}{\partial C} > 0$, $\frac{\partial^2 u}{\partial C^2} < 0$) while waste is a bad with opposite properties ($\frac{\partial u}{\partial M} < 0$, $\frac{\partial^2 u}{\partial M^2} > 0$).

Society uses output (Y) for consumption and abatement investment (R). Output Y is obtained employing a resource X, which, by assumption, is given and renewable. It follows that output also is exogenously given. Consumption and abatement investment use a part of this output so that the following constraint is satisfied

$$Y = C_t + R_t \tag{2}$$

Gross waste is a by-product of consumption goods according to the following expression

$$E_t = hC_t, 0 < h < 1 \tag{3}$$

where h measures the marginal propensity to pollute.

Waste M_t accumulates at the net rate:

$$\frac{dM}{dt} = hC_t - sA_t - vM_t \tag{4}$$

The waste stock M_t is increased by the emission hC_t, and is decreased by the environmental services sA_t provided by the abatement stock A_t devoted to clean up the environment. Further, waste accumulation is reduced by the natural biological decay, vM_t. Using (2), equation (4) can be rewritten as:

$$\frac{dM}{dt} = h(Y - R_t) - sA_t - vM_t$$

which shows the trade-off between abatement investment and consumption.

Finally, the dynamics of the abatement stock A_t chosen by the society is

$$\frac{dA}{dt} = R_t - \delta A_t \tag{5}$$

where δ is the depreciation rate of A_t.

As time periods are linked together through the stock-flow relationships, efficient waste and abatement stock must be derived from an intertemporal analysis. We proceed assuming that the society aims at maximizing the discounted social utility over some suitable time horizon. The dynamic optimization problem can be stated as follows: Select values for the control variables C_t and R_t for $t = 0, \ldots, \infty$ so as to maximize the functional

$$W = \int_0^\infty u(C_t, M_t)e^{-\rho t}dt \tag{6}$$

where ρ is the constant social discount rate subject to

$$\frac{dM}{dt} \equiv \dot{M} = h(Y - R_t) - sA_t - vM_t \tag{7}$$

$$\frac{dA}{dt} \equiv \dot{A} = R_t - \delta A_t \tag{8}$$

Associated with each state variables there is a shadow price: λ_t for the waste stock M_t and ψ_t for the abatement stock A_t.

This problem is particularly simple to analyze when the welfare function is additively separable (Plourde, 1972; Smith, 1972; Van der Ploeg and Withagen, 1991; Withagen, 1995). Therefore, we assume that

$$u(C_t, M_t) = u(C_t) + u(M_t) \equiv C_t^{\alpha} - M_t^{\gamma} \tag{9}$$

or

$$u(R_t, M_t) = (Y - R_t)^{\alpha} - M_t^{\gamma} \tag{10}$$

with $0 < \alpha < 1$ and $\gamma > 1$.

2.1 Optimality

The current valued Hamiltonian takes the form

$$H = \left\{ \begin{array}{c} [(Y - R_t)^{\alpha} - (M_t)^{\gamma}] + \lambda_t [h (Y - R_t) - sA_t - vM_t] + \\ +\psi_t (R_t - \delta A_t) \end{array} \right\} e^{-\rho t} \tag{11}$$

The transversality conditions are

$$\lim_{t\to\infty} e^{-\rho t} \lambda_t M_t = 0 \qquad \text{and} \qquad \lim_{t\to\infty} e^{-\rho t} \psi_t A_t = 0$$

In this formulation we have only the control variable R_t. The constraint $C_t = Y - R_t$ determines the flow of consumption in t.

Ignoring time subscripts, the optimal solution must satisfy the following conditions:

$$R = Y - \left(\frac{\psi - h\lambda}{\alpha}\right)^{\frac{1}{\alpha - 1}} \tag{12}$$

with

$$\dot{\psi} = \lambda s + \psi (\delta + \rho) \tag{13}$$

$$\dot{\lambda} = \gamma M^{\gamma-1} + \lambda (v + \rho) \tag{14}$$

Summing up, we have the following system of four differential equations:

$$\left\{ \begin{array}{l} \dot{\psi} = \psi (\delta + \rho) + s\lambda \\ \dot{A} = Y - \left(\frac{\psi - h\lambda}{\alpha}\right)^{\frac{1}{\alpha-1}} - \delta A \\ \dot{\lambda} = \lambda (v + \rho) + \gamma M^{\gamma-1} \\ \dot{M} = h \left(\frac{\psi - h\lambda}{\alpha}\right)^{\frac{1}{\alpha-1}} - sA - vM \end{array} \right. \tag{15}$$

where $C_t = \left(\frac{\psi - h\lambda}{\alpha}\right)^{\frac{1}{\alpha-1}}$.[1] This is a system of four differential equations with four variables as a function of time: ψ, λ, A, M.

[1] Note that the constraint is always satisfied since $Y = C + I$ that is $Y = \left(\frac{\psi - h\lambda}{\alpha}\right)^{\frac{1}{\alpha-1}} + Y - \left(\frac{\psi - h\lambda}{\alpha}\right)^{\frac{1}{\alpha-1}}$.

2.2 Properties of the optimal condition

Equation (12) can be rewritten as

$$\alpha C^{\alpha-1} = \psi - h\lambda \tag{16}$$

In this form it states that in each period the optimal social consumption C will be one in which the marginal utility of consumption $\alpha C^{\alpha-1}$ must be equal to the total value of abatement investment, $\psi - h\lambda$. The easiest way to understand this rule is to note that the optimal consumption implies that small reallocations of consumption must leave social welfare unchanged, so that the loss in utility must be equal to the increase in utility derived from abatement investment. In this view, ψ is the social value of the resource employed to produce an additional unit of abatement capital, while $h\lambda$ is the shadow social value of waste stock. Thus, condition (16) states that the value of a marginal unit of consumption must equal its marginal social value $\psi - h\lambda$.

Further, notice that written in the form $R = Y - \left(\frac{\psi - h\lambda}{\alpha}\right)^{\frac{1}{\alpha-1}}$. equation (12) states that for a society the optimal abatement investment depends on the social cost $\psi - h\lambda$ of an additional unit of abatement resource. Therefore, the expression $\psi - h\lambda$ captures the trade-off between abatement activities and polluting consumption.

2.3 Properties of optimal paths of shadow values

2.3.1 Waste

To complete the analysis of waste accumulation, let's focus on equation (14). It describes how the shadow social cost λ of waste moves along the optima path. It is equals to the present value of the loss of welfare arising from the a marginal unit of waste plus the present value of its variation. Dividing both sides by λ, we get

$$\rho + v = -\gamma \frac{M^{\gamma-1}}{\lambda} + \frac{\dot{\lambda}}{\lambda} \tag{17}$$

which states that the social cost of waste $-\gamma \frac{M^{\gamma-1}}{\lambda} + \frac{\dot{\lambda}}{\lambda}$ must be equal to its user cost, $\rho + v$. Since $\rho + v > 0$ the sign of the shadow price is negative, $\lambda < 0$. Of course, this result reflects the disutility of M.

Consider a society that bears $M = 1$ units of waste. Recall that the shadow value of waste is λ, and consider the society's choice between decreasing the waste and continuing to endure it. The disutility of waste stock is $-\gamma M^{\gamma-1}\lambda$ per unit of time. On the other hand, the stock of waste provides three distinct flows to the society. First, maintaining the stock of waste society gives up the gains it would receive by reducing the waste and enjoying the social proceeds. This has a social cost of $\rho\lambda$ per unit time. Second, waste decreases by natural decay so the society losses $v\lambda$ per unit time keeping its stock constant over time. And third, the shadow value of waste changes over time. This variation has a cost of $-\dot{\lambda}$ per unit time. Putting these three components together yields the previous social user cost of waste. Note that if $\rho + v \lesseqgtr -\gamma\frac{M^{\gamma-1}}{\lambda} + \frac{\dot{\lambda}}{\lambda}$ – that is if the social user cost of waste is smaller (greater) than the marginal disutility of waste plus its cost change – a society can find it optimal to increase (decrease) the waste stock. In steady state we have $\lambda = -\gamma\frac{M^{\gamma-1}}{\rho+v}$, that is the shadow price of waste equals the present value of social disutility.

2.3.2 Abatement capital

Finally, equation (13) provides the motion of the marginal value of abatement capital A. It can be rewritten as

$$\psi = -\frac{s\lambda}{\delta + \rho} + \frac{1}{\delta + \rho}\dot{\psi} \tag{18}$$

Intuitively, condition (18) states that for a society the internalization of the social cost λ affects directly the social ψ value of the abatement capital.

It is remarkable that the condition governing the optimal programme of society's abatement investment with waste emission should be so simple in form. The marginal social value ψ of the abatement investment is equal to the discounted gross value λ of an additional unit of waste plus the discounted value of its capital gain $\dot{\psi}$. Thus, one important implication of our model is that the marginal ψ value captures all information relevant to the society's abatement investment decision. It is also remarkable that in the present model the state and costate variables affect each other mutually: the value ψ of the abatement capital depends on the social cost of waste λ, and the flow of emissions $\dot{M}$ depends on the abatement stock A_t whose value is affected by the charge λ. This complex relationships feeds the economic-ecological functioning of the model, and shapes the trade-off between waste accumulation and abatement capital.

2.3.3 Internalization vs externality

If p_c is the social price of a unit of consumption net of its polluting cost τ, then $P_c = p_c + \tau$ is the gross price of the consumption good. Therefore, P_c is the social price of consumption, including the cost of emission, and in equilibrium it cannot differ from the marginal utility of consumption plus the marginal disutility of emission, that is $P_c = p_c + \tau = \alpha C_t^{\alpha-1} + \lambda h$. From condition (16) we know that in equilibrium the marginal utility of consumption must be equal to the marginal shadow value ψ of producing an additional unit of abatement capital minus the marginal shadow cost of waste λh, that is

$$\psi = p_c + \tau \tag{19}$$

Hence, condition (16) can be rewritten as

$$\alpha C^{\alpha-1} = p_c + \tau - h\lambda \tag{20}$$

In this equation τ is a fee which internalizes the cost of polluting consumption, while $-h\lambda$ measures the marginal social utility of waste internalization. The higher is the corrective tax

τ, the smaller is consumption and the higher the corresponding marginal utility. Equilibrium requires that $\tau - h\lambda > \tau$, so that consumption can include the social opportunity cost.

Using the same argument, we could define $P_R = p_R + \sigma$ as the gross price of a unit of abatement capital, comprehensive of its market price p_R and shadow value σ. However, given the market failure, $p_R = 0$ so that the gross price of the abatement capital is simply $P_R = \sigma$. Since the marginal disutility of consumption in term of waste is h, then $1/h$ is the marginal disutility of waste in term of abatement investment. In equilibrium it must be verified that $\sigma = \frac{1}{h}\left(\psi - \alpha C^{\alpha-1}\right)$, that is the social price σ of an additional unit of abatement capital must be equal to the social disutility provided by the marginal value of an additional unit of abatement capital net of its social cost in terms of consumption. Hence, condition (12) can be rewritten as

$$R = Y - \left(\frac{\psi - h\sigma}{\alpha}\right)^{\frac{1}{\alpha-1}} \tag{21}$$

This implies that for optimality

$$-\lambda = \sigma \tag{22}$$

The value σ can be seen as a transfer paid by society to subsidize additional units of abatement capital. Equilibrium requires $\psi - h\sigma > h\sigma$, so that the abatement investment can includes the social opportunity cost.

To sum up, in the present model τ can be seen as a Pigouvian tax per unit of waste, and σ as a subsidy to promote abatement investment. Therefore, the socially optimal allocation can be sustained by levying a tax τ on consumption and redistributing the revenues σ to build up abatement capital.

A further implication of the previous analysis is the following. If τ is equal to zero, it may be the case that no private action is expected which will reduce the consumption and stimulate society in producing abatement capital. In this case, $Y = C$, $A = 0$ and $M = hY/v$. Therefore, the market allocation does not internalize the stock of waste and leads to a private solution which is sub-optimal. In fact, the market solution will not refrain from excessively high consumption which produces waste along the way.

Obviously, when $Y = C$ the abatement capital is not supplied. Indeed, since abatement capital is not marketable, no private agent will be engaged in this activity unless some authority will pay the agent the social price σ to dispose waste. As said above, Pigouvian taxes τ can be imposed to sustain a tax-cum-subsidy scheme. As demonstrated in Plourde (1972), this scheme can produce a transition toward a new steady state which internalize the cost of waste. Using our symbols, the social steady state requires that $\dot{\psi} = \dot{A} = \dot{\lambda} = \dot{M} = 0$, and the governmental budget of taxes and subsidies will be in balance.

3 Analyzing the dynamic system

Before proceeding, we determine the steady values of our four variables. Setting $\dot{\psi} = \dot{A} = \dot{\lambda} = \dot{M} = 0$ and denoting the steady state values with an asterisk, the steady state of the economic-ecological system (15) $S^* = (\psi^*, A^*, \lambda^*, M^*)$ is defined by the following system of equations

$$\begin{cases} \psi^* = -\frac{s}{\rho+\delta}\lambda^* \\ A^* = \frac{1}{\delta}\left[Y - \left(\frac{\psi^*-h\lambda^*}{\alpha}\right)^{\frac{1}{\alpha-1}}\right] \\ \lambda^* = -\frac{\gamma}{\rho+v} M^{*\gamma-1} \\ M^* = \frac{h\left(\frac{\psi^*-h\lambda^*}{\alpha}\right)^{\frac{1}{\alpha-1}} - sA^*}{v} \end{cases} \tag{23}$$

Now, we study local dynamics of system (15). For this purpose, let's linearize the dynamic system (15) around the steady state S^*

$$\dot{S} = J\left(S - S^*\right) \tag{24}$$

where J is the Jacobian

$$\begin{pmatrix} \delta+\rho & 0 & s & 0 \\ G & -\delta & -hG & 0 \\ 0 & 0 & \nu+\rho & \gamma\left(\gamma-1\right)M^{*\gamma-2} \\ -hG & -s & h^2G & -\nu \end{pmatrix} \tag{25}$$

and $G = \frac{1}{\alpha(1-\alpha)}\left(\frac{\psi^*-h\lambda^*}{\alpha}\right)^{\frac{1}{\alpha-1}-1}$. The linearization gives rise to a characteristic equation in x of fourth degree of the form

$$Q\left(x\right) = x^4 + bx^3 + cx^2 + dx + e = 0 \tag{26}$$

where

$$\begin{aligned} b &= -2\rho, \\ c &= \rho^2 - \rho\left(\delta+v\right) - \left(\delta^2+v^2+\Sigma\right), \\ d &= \rho^2\left(\delta+v\right) + \rho\left(\delta^2+v^2+\Sigma\right), \\ e &= \left(\rho+v\right)\left[-\delta\left(-v\left(\rho+v\right)-\Sigma\right) + \frac{s}{h}\Sigma\right] + \frac{s}{h}\Sigma\left(\delta+\frac{s}{h}\right) \end{aligned}$$

and $\Sigma = \gamma\left(\gamma-1\right)M^{*\gamma-2}h^2G > 0$. Appendix A computes the four roots of the quartic equation (26) and solves the problem of optimal welfare with waste and abatement capital stocks.

4 Transitional dynamics: Subsidies and taxes

Since we are dealing with a system in four dimensions, it is difficult to understand what exactly is happening outside the steady state. In order to make the matter tractable, we will proceed by separating the system in two blocks. In the first, we will analyze the subsystem composed of the dynamics of the abatement capital A_t and its shadow price ψ_t, assuming all other variables constant at their steady states values. In the second, we will analyze the dynamics of waste capital M_t and its shadow price λ_t, supposing constant all other variables constant at their steady states values.

Focusing on the subsystems, we can address many issues of environmental policies. For example, we may suppose that society adopts a subsidy to stimulate the realization of additional units of abatement capital A. Alternatively, society can manage the Pigouvian taxes to penalize polluting consumption.

4.1 The dynamics of A and ψ

To begin with, consider the dynamics of the abatement capital A and of its shadow price ψ, given the steady state values λ^* and M^*. The equation of motion of these two variables are:

$$\begin{cases} \dot{\psi} = \psi\left(\delta + \rho\right) + s\lambda^* \\ \dot{A} = Y - \left(\frac{\psi - h\lambda^*}{\alpha}\right)^{\frac{1}{\alpha-1}} - \delta A \end{cases} \tag{27}$$

Linearizing the sub system around the steady state values A^* and λ^*, we get:

$$\begin{cases} \dot{\psi} = \psi\left(\delta + \rho\right) + s\lambda^* \\ \dot{A} = Y - \left(\frac{\psi - h\lambda^*}{\alpha}\right)^{\frac{1}{\alpha-1}} + G\left(\psi - \psi^*\right) - \delta A \end{cases} \tag{28}$$

Since the Jacobian of the system (28)

$$\begin{pmatrix} \delta + \rho & 0 \\ G & -\delta \end{pmatrix} \tag{29}$$

is negative, we are in presence of a saddle point.

4.2 Subsidies

In this section we examine the implications of changes in subsidies. As said above, the subsidy affects the value of the abatement capital since $\lambda = -\sigma$. By now, recall that in steady state the shadow price ψ takes the value

$$\psi^* = -\frac{s}{\rho + \delta}\lambda^* \tag{30}$$

where $\lambda^* < 0$ so that $\psi^* > 0$. Therefore, a rise in σ affects negatively λ^* and rises ψ^*, stimulating society to enlarge the stock of abatement capital. In the phase diagram a rise in σ shifts the $\dot{\psi} = 0$ locus up, moving the economy to a new long run equilibrium. Since for a given capital stock A rents are higher, smaller capital gains are now needed to demand additional units of abatement capital. From our analysis of phase diagram, we know the

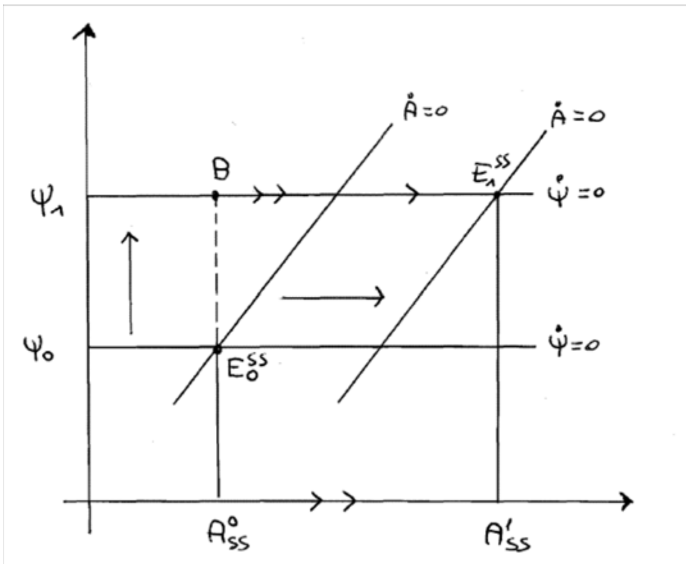

Figure 1: The effects of a permanent subsidy

effect of a higher subsidy: ψ jumps immediately on the new saddle path for a given capital stock. Then, A moves along the path to the new steady state

Assume that the change in subsidy is *permanent.* Such a change shifts up the locus $\dot{\psi} = 0$. Similarly, along the locus $\dot{A} = 0$ the effect of a rise in σ increases λ^* moving the isocline to the right since $\partial A/\partial\sigma > 0$. The phase diagram, therefore, looks as the one drawn in figure 1.

Figure 1 states that initially society is doing replacement investment just sufficient to maintain its abatement capital stock A_0^{ss}. Following the rise in σ, the marginal ψ value and the corresponding investment rises instantaneously to a new higher level, causing the abatement capital stock to gradually rise toward the new steady state level A_1^{ss}. But, the only path which leads to the new steady state, with no future jumps in ψ, is the new stable path. Therefore, ψ jumps instantaneously to ψ_1 and thereafter moves along the stable path.

The intuition behind this result is straightforward. A permanent subsidy σ raises the abatement capital stock chosen by the society. However, in the short run since the abatement capital stock cannot be adjusted instantaneously, because of the implicit cost of the reallocation of consumption over time, existing abatement capital earns rents and its shadow value ψ rises strongly. The higher shadow value stimulates society to enlarge the abatement capital stock which converges to the new steady state value A_{SS}^1 in the long run.

Now, consider an increase in subsidy that is known to be *temporary.* This case is illustrated in figure 2 where σ rises in the short run, but eventually will come back to its initial level. Assume that the economy begins in the steady state E_0^{SS}. After the subsidy increase, there is an upward shift of the marginal ψ; then, the economy anticipates that ψ will return to its initial level at some later time T. The key insight needed to find the effect of this change

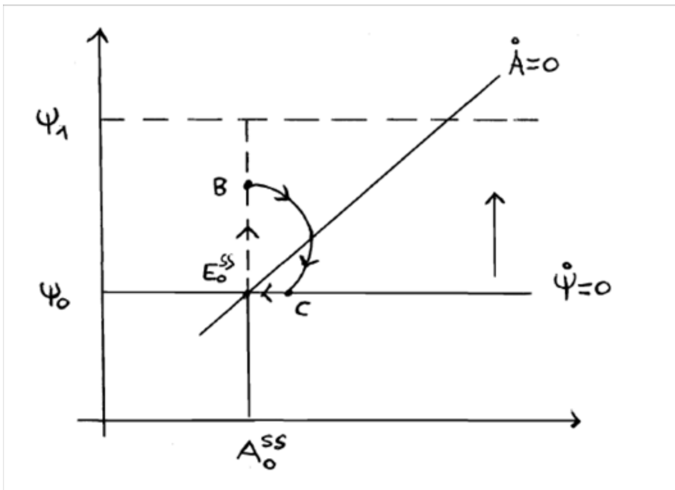

Figure 2: The effect of a temporary increase in subsidy

is that there cannot be an anticipated jump in ψ. Therefore, at time T, both the variables A and ψ must be on the saddle path leading back to the initial long-run equilibrium: if they were not, ψ would have to jump to get back to its long-run equilibrium. But ψ is a continuos function and cannot jump between t_0 and T.

Together these facts tell us how abatement investment responds to the temporary subsidy. ψ jumps immediately to the point B such that, given the initial abatement capital, ψ and A reach the old saddle path at exactly time T. This is shown in figure 2 where ψ jumps from point E_0^{SS} to point B when the subsidy is adopted; then, they move gradually to point C, arriving there at time T. Finally, they move to the old steady state.

This analysis has several implications. First, temporary subsidy rises abatement investment in the short and medium run. Second, comparing figures 1 and 2 emerges that ψ rises less than it does if the increase in subsidy is permanent, so that investment responds less. Finally, figure 2 shows that the path of A and ψ crosses the locus $\dot{A} = 0$ before it reaches the old saddle path. Therefore, abatement capital stock begins to decline before temporary subsidy return to initial level (along the old saddle path). Indeed, when only a brief period of high subsidy remains, the society finds optimal to reduce investment immediately since it is too costly to adjust capital stock later. The main outcome of this analysis is that temporary subsidies do not change the long-run equilibrium of the economy. In other words, it is not just current subsidies, but their entire path over time that affect abatement capital and waste accumulation.

4.3 The dynamics of M and λ

In this section we consider the system dynamics of the waste capital and its shadow cost. The equation of motion of these two variables are:

$$\dot{\lambda} = \lambda(\rho + v) + \gamma M^{\gamma-1} \tag{31}$$

$$\dot{M} = h\left(\frac{\psi^* - h\lambda}{\alpha}\right)^{\frac{1}{\alpha-1}} - sA^* - vM \tag{32}$$

Recall that the variables of the other subsystem, A and ψ, are set equal to their steady state values. Linearizing the system around the steady state values M^* and λ^*, we get:

$$\begin{aligned}
\dot{\lambda} &= \lambda(v+\rho) + \gamma(M^*)^{\gamma-1} + \gamma(\gamma-1)(M^*)^{\gamma-2}(M - M^*) \\
\dot{M} &= h\left(\frac{\psi^* - h\lambda^*}{\alpha}\right)^{\frac{1}{\alpha-1}} - sA^* + h^2 G(\lambda - \lambda^*) - vM
\end{aligned}$$

Since the Jacobian of this system

$$\begin{pmatrix} \rho + v & \gamma(\gamma-1)(M^*)^{\gamma-2} \\ h^2 G & -v \end{pmatrix}$$

is negative, we are in presence of a saddle point.

4.4 Taxes

In a decentralized economy, no market will exist to reflect the social cost of waste τ. The steady state of the private solution is therefore obtained by setting $\lambda = p_c$ in the control problem. If p_c is sufficiently small, in the extreme case $p_c = 0$, starting at the initial level M_0, the decentralized stock of waste will grow at the rate $\frac{dM}{dt} = hC_t - vM_t > 0$ until $M = M^P$. This is shown in figure 3. However, M^P is not an optimum. For a social optimum, abatement capital is necessary and this occurs at the point E_0 (the social steady state) where $M_0^{SS} < M^p$ with an appropriate waste charge λ_0.

Starting at the waste stock M_0^{SS} assume now that the tax τ rises *permanently*. The isocline for $\dot{\lambda} = 0$ does not change, while the isocline $\dot{M} = 0$ shifts upwards by τ because in equilibrium $\psi^* = p_c + \tau$. This is shown in figure 3. If the tax increase is permanent, λ jumps up from E_0^{SS} to B to the new saddle path at the time it is announced. Intuitively, because the corrective tax reduces consumption in the long run, it means that polluting emissions will be lower, and thus that existing waste is less valuable. M and λ then move together along the saddle path to the new long run equilibrium M_1^{SS}.

Consider instead a *temporary* rise of tax τ and its effect on waste. The initial effect of this change is to shift the isocline $\dot{\lambda} = 0$ on the left. When an increase in tax is known to be temporary, it is also known that the system will return to its initial path at some later time T. An example of this dynamics is illustrated in figure 4.

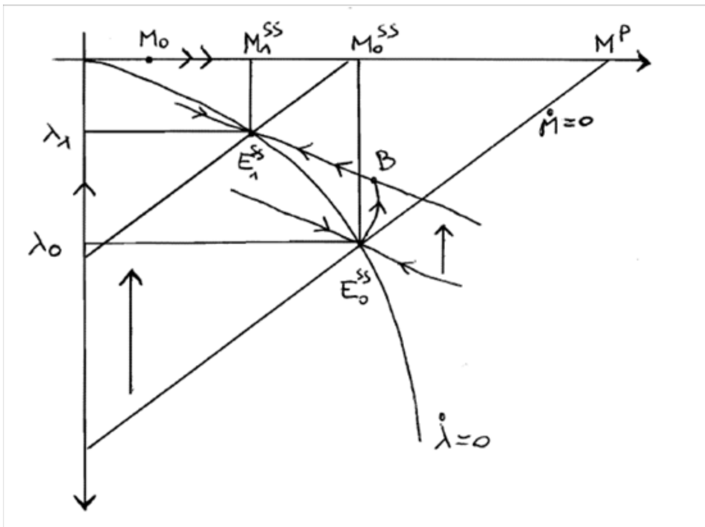

Figure 3: A permanent change in tax

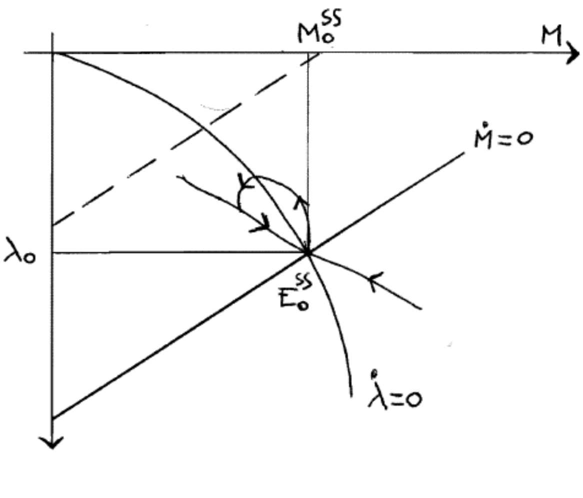

Figure 4: The effect of a temporary change in tax

The temporary increase of a corrective tax causes waste stock to fall to a point where the dynamics of M and λ bring them back to the old saddle path just as the variation of tax τ expires. They will then converge back to the initial steady state E_0^{SS}. As the figure 4 shows, λ does not fall all the way to its value on the new saddle path. Therefore, a temporary increase of τ decreases the social cost of waste less than a similar permanent tax does. The reason is that because the temporary tax does not lead to a permanent decrease in consumption, it causes a smaller reduction in the social cost associated to the existing waste stock.

Finally, note that the figure (4) shows that under the temporary tax, the social cost λ of waste is rising in the later part of the period that the increase of tax is in effect, even when waste is decreasing. Therefore, after a point, the corrective tax leads to a growing social cost and, eventually, to a growing waste accumulation before corrective tax returns to normal. This happens because society anticipates that future tax will decrease, thus rising future consumption. As before, this result implies that it is not just current Pigouvian tax but its entire path over time that affects waste and investment accumulation.

5 Waste control and the Ramsey problem

Up to now we have supposed that output is exogenous. In the following, we will endogenize output by introducing an aggregate production function $Y_t = f(K_t)$, where K_t is the stock of capital. Output has now three different uses: it can be either consumed, invested in abatement capital or accumulated in productive capital. That is,

$$Y_t = C_t + R_t + \dot{K}_t + \omega K_t \tag{33}$$

where $\dot{K}$ is net investment and ω is the rate of depreciation. That said, the rest of the model remains unchanged. The current valued Hamiltonian takes the form

$$H = \left\{ \begin{array}{c} (C_t)^{\alpha} - (M_t)^{\gamma} + \lambda_t (hC_t - sA_t - vM_t) + \\ +\psi_t (R_t - \delta A_t) + \mu_t [f(K_t) - C_t - R_t - \omega K_t] \end{array} \right\} e^{-\rho t} \tag{34}$$

and from the first order conditions we get

$$C = \left(\frac{\mu - h\lambda}{\alpha} \right)^{\frac{1}{\alpha - 1}} \tag{35}$$

and

$$\psi = \mu \tag{36}$$

which implies that the shadow price for investment is the same independently of its alternative use in abatement activities or in production. Further, condition (36) also implies that $\psi = \alpha C^{\alpha-1} + \lambda h$ which restates the optimal consumption rule (16) when output Y in endogenous.

The equations for the costate variables are

$$\dot{\psi} = \lambda s + \psi (\delta + \rho) \tag{37}$$

$$\begin{aligned} \dot{\lambda} &= \gamma M^{\gamma-1} + \lambda (v + \rho) && (38) \\ \dot{\mu} &= \mu [\rho + \omega - f'(K)] && (39) \end{aligned}$$

Summing up, we have the following system of six differential equations:

$$\begin{cases} \dot{\psi} = \psi (\delta + \rho) + s\lambda \\ \dot{A} = R - \delta A \\ \dot{\lambda} = \lambda (v + \rho) + \gamma M^{\gamma-1} \\ \dot{M} = hC - sA - vM \\ \dot{\mu} = \mu [\rho + \omega - f'(K)] \\ \dot{K} = f(K) - C - R - \omega K \end{cases} \tag{40}$$

Such a problem is hard to solve analytically. But, due to the structure of this problem, we get a closed form solution. In order to keep the analysis manageable, we reduce these system of six differential equations by remembering that $\psi = \mu$ so that $\dot{\psi} = \dot{\mu}$ at any instant of time. In addition, following the intuition by Van der Ploeg and Withagen (1991), we define a new variable T as the total stock of capital, $K + A$, so that $\dot{T} = \dot{K} + \dot{A}$. Using these notations, we can rewrite the system (40) as:

$$\begin{cases} \dot{\psi} = \psi (\delta + \rho) + s\lambda \\ \dot{T} = f(K) - C - \delta T \\ \dot{\lambda} = \lambda (v + \rho) + \gamma M^{\gamma-1} \\ \dot{M} = hC - sA - vM \\ \dot{\mu} = \mu [\rho + \omega - f'(K)] \\ \dot{K} = f(K) - C - R - \omega K \end{cases} \tag{41}$$

where without loss of generality we assume that $\omega = \delta$. Now, to eliminate R we rewrite the equation of motion of capital stock by using the first order condition for investment in abatement capital. Since $\dot{\psi} = \dot{\mu}$, it follows from the previous system that

$$-\mu f'(K) = s\lambda, \text{ or } \frac{\mu}{\lambda} = -\frac{s}{f'(K)} \tag{42}$$

By taking logs of both sides, differentiating with respect to time and equating, we get

$$\begin{aligned} \frac{\dot{\mu}}{\mu} - \frac{\dot{\lambda}}{\lambda} &= \rho + \omega - f'(K) - (\rho + v) - \frac{\gamma M^{\gamma-1}}{\lambda} \\ &= \frac{d}{dt} \ln \left(-\frac{s}{f'(K)} \right) = \frac{d}{dt} \ln \frac{1}{f'(K)} = \frac{f''(K)}{f'(K)} \dot{K} \end{aligned} \tag{43}$$

that is

$$\dot{K} = -\frac{f'(K)}{f"(K)}\left(f'(K) - \omega + \frac{\gamma M^{\gamma-1}}{\lambda} + v\right) \tag{44}$$

which describes the optimal path of K as a function of the marginal productivity of capital and its variation, and of the marginal disutility of accumulated waste. We label this optimal intertemporal equation for the productive capital as *Environmental-Keynes-Ramsey rule.*

5.1 An Environmental-Keynes-Ramsey rule

To explain the economic meaning of condition (44), recall that the costate variable λ of the waste stock is a negative shadow value, i.e., it is a cost. As discussed above, to internalize this cost, it must be verified that $\lambda = -\sigma$. Substituting in condition (44), we get:

$$\dot{K} = -\frac{f'(K)}{f"(K)}\left[f'(K) - \omega - \frac{\gamma M^{\gamma-1}}{\sigma} + v\right] \tag{45}$$

Also, let's assume that the production function has form $f(K) = K^{\varepsilon}$, where $\varepsilon < 1$. We can now rewrite equation (45) as

$$\frac{\dot{K}}{K} = \frac{1}{1-\varepsilon}\left[K^{\varepsilon-1} - \omega - \left(\frac{\gamma M^{\gamma-1}}{\sigma} - v\right)\right] \tag{46}$$

Capital accumulation is positive if and only if the net marginal product of capital $K^{\varepsilon-1} - \omega$ is greater than the net social cost it generates, given by the marginal disutility of waste weighted by its shadow cost minus the biological decay. This difference is in turn weighted by the inverse of the elasticity of the capital marginal product $\frac{1}{1-\varepsilon}$. As said, this condition resembles the famous Keynes-Ramsey rule for consumption where the weighting factor is the reciprocal of marginal utility of consumption (i.e., the elasticity of intertemporal substitution) and the difference is between the marginal product of capital and the rate of time preference (Ramsey, 1928; Blanchard and Fischer, 1989).

The source of this result is that K affects indirectly the welfare function through its dynamic relationship with consumption, abatement capital and waste emissions period by period. Indeed, when K is relatively low, aggregate output, consumption and waste emissions are relatively low. Therefore, since the associated marginal value $K^{\varepsilon-1} - \omega$ is relatively high, for a given waste stock M, the growth rate $\frac{\dot{K}}{K}$ is also relatively high.

Equation (46) states that $\frac{\dot{K}}{K}$ equals the difference between $K^{\varepsilon-1} - \omega$ and $\frac{\gamma M^{\gamma-1}}{\sigma} - v$. The first term is a downward-sloping curve. The second term is an upward-sloping curve since $\gamma > 1$, so that M rises when Y and K rise, and consequently C rises as well. Figure 5 is the graphical counterpart of this equation and shows that the vertical distance between the two curves equals the growth rate of capital, and the crossing point E_0^{SS} corresponds to the steady state K^{SS}.

From the inspection of figure 5 emerges that to the left of the steady state the curve $K^{\varepsilon-1} - \omega$ lies above $\frac{\gamma M^{\gamma-1}}{\sigma} - v$. Hence, the growth rate of K is positive and the productive

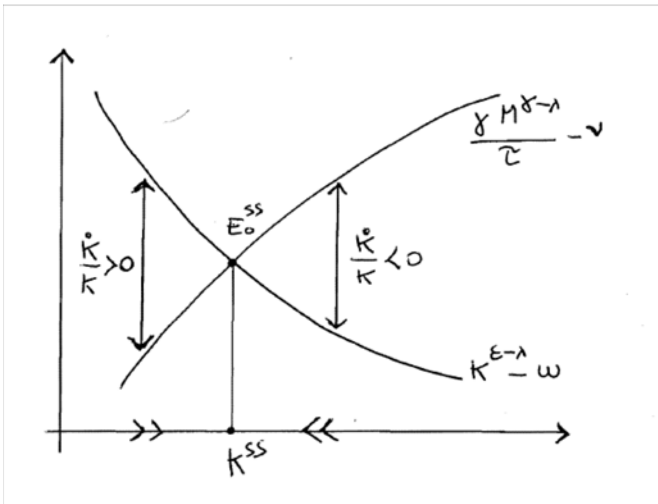

Figure 5: The growth rate of K

capital rises over time. As K increase, $\frac{\dot{K}}{K}$ declines and approaches zero as capital approaches the steady state K^{SS}. Two are the sources of this result: the diminishing returns to capital and the rising disutility of the waste stock.

An analogous argument demonstrates that if economy starts to the right of K^{SS} then the growth rate is negative and K falls over time. Thus, the economic-ecological system is locally asymptotically stable. Note however that under the assumption of diminishing returns an infinite productive capital growth is not sustainable over time. Further, note that, since the waste stock reduces the social value of the productive capital, the higher is the perceived disutility of M, the smaller will be the growth rate of capital along the transition, and the smaller the corresponding capital stock in the steady state. Therefore, the internalization of waste stock affects *both* the growth rates and the levels of the economy. In other words, the endogenous relationship between waste, abatement and productive stocks – if properly managed by society – can allow to control optimally waste accumulation and drive the economy toward a sustainable path.

Finally, remember that the society can use taxes and subsidies to change the steady state. Imagine society decides to rise permanently subsidies σ in order to promote abatement capital A per unit of waste. What happens to productive capital stock K? This choice reduces the social disutility of waste, shifting the schedule $\frac{\gamma M^{\gamma-1}}{\sigma} - v$ to the right. Hence, the intersection with the schedule $K^{\varepsilon-1} - \omega$ also shifts to the right, and the new steady state capital stock exceeds the old one. How does economy adjust from one equilibrium to the other? At the initial steady state K^{ss}, the gap between the two curves is now positive, and the net social marginal return from productive capital is higher than the social disutility of waste. This generates an increase in K. The result, therefore, is that a permanent rise of subsidies, levied to promote abatement capital, generates temporarily positive growth rates of K. In the long-run, the levels of A and K are permanently higher, but their growth rates return to zero. For the same reason, the level of C rises in the long-run, but its growth rate follows K and A during transitional phase and converges to zero in the steady state. Thus, changes in these kind of policies have *temporary* effects on the growth rates of consumption, abatement and productive capital stocks and *permanent* effects on their levels.

In the next section, we shall employ this condition for capital accumulation in place of the identity between production and its uses, that way eliminating the unknown R which will be determined residually.

6 The steady state

As before, we will begin the analysis by determining the steady values S^* of our variables. Denoting again the steady state values with an asterisk, they can be derived as the solution of the following system of equations

$$\begin{cases} \psi^* = -\frac{s}{\rho+\delta}\lambda^* \\ A^* = \frac{1}{\delta}R^* \\ Y^* = \left(\frac{\psi^*-h\lambda^*}{\alpha}\right)^{\frac{1}{\alpha-1}} + R^* + \omega K^* \\ \lambda^* = -\frac{\gamma}{\rho+v}M^{*\gamma-1} \\ M^* = \frac{h\left(\frac{\psi^*-h\lambda^*}{\alpha}\right)^{\frac{1}{\alpha-1}} - sA^*}{v} \\ \rho+\omega = \varepsilon\left(K^*\right)^{\varepsilon-1} \end{cases} \tag{47}$$

Once determined A^* and K^*, we know that $T^* = K^* + A^*$.

We now proceed by linearizing the system around its steady state. Precisely, we write

$$\begin{cases} \dot{\psi} = \psi\left(\delta+\rho\right) + s\lambda \\ \dot{T} = K^{\varepsilon} - C - \delta T \\ \dot{\lambda} = \lambda\left(v+\rho\right) + \gamma M^{\gamma-1} \\ \dot{M} = hC - s\left(T-K\right) - vM \\ \dot{\mu} = \mu\left[\rho+\omega-f'\left(K\right)\right] \\ \dot{K} = \frac{1}{1-\varepsilon}\left(K\varepsilon K^{\varepsilon-1} - K\gamma\frac{M^{\gamma-1}}{\tau} + \left(v-\omega\right)K\right) \end{cases} \tag{48}$$

Calculations give the following Jacobian matrix of the linearized system:

$$J = \begin{pmatrix} \delta+\rho & 0 & s & 0 & 0 \\ G & -\delta & -hG & 0 & \rho+\delta \\ 0 & 0 & \nu+\rho & \gamma\left(\gamma-1\right)M^{*\gamma-2} & 0 \\ -hG & -s & h^2G & -\nu & s \\ 0 & 0 & \frac{1}{1-\varepsilon}\frac{K^*}{\lambda^*}\left(\rho+v\right) & -\frac{\gamma-1}{1-\varepsilon}\frac{K^*}{M^*}\left(\rho+v\right) & \rho+\omega \end{pmatrix} \tag{49}$$

where the last two rows and columns refer to K, and where use has been made of the fact that in steady state $\rho + v = -\frac{\gamma}{\lambda^*}M^{*\gamma-1}$ and $\rho + \omega = \varepsilon\left(K^*\right)^{\varepsilon-1}$.

6.1 The characteristic equation

The characteristic equation for the linearized system is

$$\begin{vmatrix} \delta+\rho-x & 0 & s & 0 & 0 \\ G & -\delta-x & -hG & 0 & \rho+\delta \\ 0 & 0 & \nu+\rho-x & \gamma(\gamma-1)M^{*\gamma-2} & 0 \\ -hG & -s & h^2G & -\nu-x & s \\ 0 & 0 & \frac{1}{\varepsilon-1}\frac{K^*}{\lambda^*}(\rho+v) & -\frac{\gamma-1}{\varepsilon-1}\frac{K^*}{M^*}(\rho+v) & \rho+\omega-x \end{vmatrix} = 0 \tag{50}$$

which has many similarities with the one we studied in absence of output growth. In fact, equation (50) is the same of the previous problem, beside the last row and column. Therefore, we proceed by expanding to

$$\begin{aligned} &(\rho+\delta-x)\left\{(\rho+\delta-x)\left[(-\delta-x)\left[(\nu+\rho-x)(-\nu-x)-L\right]\right]+s\left[(\delta+x)\frac{L}{h}+\frac{s}{h^2}L\right]\right\} \\ &-(\delta+x)(\rho+\delta-x)N\left[\frac{(\nu+\rho-x)}{M^*}+\frac{\gamma M^{*\gamma-2}}{\lambda^*}\right] \\ &+(\rho+\delta)(\rho+\delta-x)\left[\frac{(\nu+\rho-x)}{M^*}+\frac{\gamma M^{*\gamma-2}}{\lambda^*}\right]=0 \end{aligned}$$

where as before $L = h^2G\gamma(\gamma-1)M^{*\gamma-2}$ and $N = s\frac{\gamma-1}{\varepsilon-1}K^*(\rho+v)$. Remembering that $\rho+v = -\frac{\gamma}{\lambda^*}M^{*^{\gamma-1}}$, we simplify the last expression by writing

$$\begin{aligned} (\rho+\delta-x)\{(\rho+\delta-x)\left[(-\delta-x)\left[(\nu+\rho-x)(-\nu-x)-L\right]\right]+ \\ +s\left[(\delta+x)\tfrac{L}{h}+\tfrac{s}{h^2}L\right]+Nx\tfrac{x}{M^*}-N\rho x\}=0 \end{aligned} \tag{51}$$

Hence, a root of the characteristic equation is $\rho+\delta$. The remaining four eigenvalues are the roots of the following quartic equation:

$$Q(x) = x^4+bx^3+cx^2+dx+e=0 \tag{52}$$

where b, c, d and e are given coefficients.[2] From the mathematical point of view, it is remarkable that the structure of our intertemporal problem provides a closed form solution even in presence of three state variables, namely waste, abatement capital and productive capital stocks. Appendix B computes the four roots of the quartic equation.

6.2 The dynamics of the system

Having determined characteristic roots (see Appendix B), we now focus on the dynamics of the system. Let x_1 and x_2 be the two characteristic negative and stable roots. Correspondingly, we have two characteristic vectors, v^1 and v^2:

$$Jv_1 = x_1v_1 \text{ and } Jv_2 = x_2v_2$$

[2] Precisely, $b=-2\rho$, $c=\rho^2-\rho(\delta+v)-\left(\delta^2+v^2+L+\frac{N}{M^*}\right)$, $d=\rho^2(\delta+v)+\rho\left(\delta^2+v^2+L-N\right)$, and $e=(\rho+v)\left[\delta\left(v(\rho+v)+L\right)+\frac{s}{h}L\right]+\frac{s}{h}L\left(\delta+\frac{s}{h}\right)$.

Hence, we can write the linearized version of the five differential equations as follows:

$$
\begin{aligned}
\psi(t) &= A_1 v_1^1 \exp(x_1 t) + A_2 v_1^2 \exp(x_2 t) + \psi^* \\
T(t) &= A_1 v_2^1 \exp(x_1 t) + A_2 v_2^2 \exp(x_2 t) + T^* \\
\lambda(t) &= A_1 v_3^1 \exp(x_1 t) + A_2 v_3^2 \exp(x_2 t) + \lambda^* \\
M(t) &= A_1 v_4^1 \exp(x_1 t) + A_2 v_4^2 \exp(x_2 t) + M^* \\
K(t) &= A_1 v_5^1 \exp(x_1 t) + A_2 v_5^2 \exp(x_2 t) + K^*
\end{aligned}
\tag{53}
$$

where A_1 and A_2 are two arbitrary constants. Assuming as given by history $T(0)$ and $M(0)$, we can determine them from the second and fourth equation of the previous system in the following way:

$$
\begin{aligned}
T(0) &= A_1 v_2^1 + A_2 v_2^2 + T^* \\
M(0) &= A_1 v_4^1 + A_2 v_4^2 + M^*
\end{aligned}
$$

Then, the dynamics of $A(t)$ is calculated residually since $A(t) = T(t) - K(t)$.

The comparative statics are left for further research.

7 Conclusions

In this paper we have addressed the control problem of a social optimum in presence of waste, abatement capital and productive capital stocks under the alternative scenarios of fixed and growing output. On the analytical ground, one of the main outcome of this contribution is that we are able to find a closed form solution to this kind of problem. The special feature of this solution lies in the fact that in both scenarios it is possible to manage the system of differential equations in order to get a quartic characteristic equation which has the property to be a biquadratic.

Waste externalities arise because waste is a by-product of consumption. When matters are left to the market it may be the case that the optimal abatement capital is equal to zero in the long-run and all output is consumed. Then, waste stock arises at its maximum level, determining a sub-optimal allocation. Society can internalize this cost levying corrective taxes and providing incentives to invest in abatement capital. Appropriate Pigouvian taxes and subsidies are used to change the private steady state, moving the economic-ecological system from the private allocation to the social one.

When output is given, the maximization of welfare requires a social charge for the abatement capital and a social cost for the waste stock. The interrelation between the two shadow values determines the social optimum. Under the assumption of given output, the closed form solution helps in analyzing the dynamic properties of the model. Since this solution is locally asymptotically stable, comparative statics can be developed, focusing on the subsystems of abatement capital or waste stocks. From the analysis emerges that in response to subsidies or taxes society tends to accumulate abatement capital allowing the stock of waste to decrease in the long-run. However, the adoption of either temporary subsidies or taxes do not change the long-run equilibrium of the economic-ecological system. Indeed, it is not just current subsidies or taxes, but their entire path over time that affects the accumulation of abatement capital and waste stock. Thus, expectations about how long the subsidies and taxes would last have a special role in affecting abatement investment and waste accumulation.

When one allows waste accumulation and abatement capital in a Ramsey model, the productive capital stock in steady state is less than the one obtained without abatement capital, and consequently consumption is also less. But, abatement capital stock is now positive and waste accumulation is reduced. Therefore, with three state variables – waste, abatement capital and productive capital – the dynamic of the system is much more complex. Nonetheless, we get an Environmental-Keynes-Ramsey rule which describes the optimal accumulation of productive capital, and hence of the output and consumption growth, as a function of economic and ecological variables. Basically, we show that when waste and abatement capital stocks are embedded in a Ramsey model, capital accumulation is positive *if and only if* the net marginal product of capital is greater than the net social cost it generates, given by the marginal disutility of waste weighted by its shadow cost. This condition resembles the well known Keynes-Ramsey rule for consumption. From the analysis of this rule also emerges that environmental policies have temporary, but not permanent, effects on growth rates of abatement and productive capital stocks.

For the models that have been used, it has been shown that charges can be levied to redirect wasting activities (as consumption) towards alternative ones optimally chosen by society. A socially optimal tax is derived as the solution of the intertemporal problem. This tax affects the disutility of consumption and the incentives to invest in abatement capital and to produce output. However, as it is clear from our discussion, variations of taxes influence not only the dynamic of the abatement capital, but also the accumulation of waste and productive capital. Hence, one important implication of our model is that waste accumulation affects the optimal path of both abatement and productive capital stocks, altering their marginal social values through the attributed social changes. Therefore, flows and stocks must be considered simultaneously to get robust assessments about the effectiveness of the environmental policies.

Based on these considerations, the problem of waste accumulation is actually much more complex than it is usually presented. Therefore, we believe that the issues raised in this paper need to be taken into account to provide a comprehensive analysis of the intertemporal problems related to waste emission and alternative resource uses. The present model is not conclusive, but it opens new perspectives concerning the efficient and optimal use of natural resources. The next step would be to introduce irreversibility and uncertainty. Further, abatement capital stock may be heterogeneous or located in such a way that costs may differ for different kind of stocks. Future research should focus on such integrated models.

Appendix A The quartic equation

Our quartic equation $Q(x) = x^4 + bx^3 + cx^2 + dx + e = 0$ has the special property that can be reduced to a biquadratic equation by a double change of variable: first, from x to $y = x + \frac{b}{4}$, and then from y to $z = y^2$.

The first change produces the following equation:

$$P(y) = \left(y - \frac{b}{4}\right)^4 + b\left(y - \frac{b}{4}\right)^3 + c\left(y - \frac{b}{4}\right)^2 + d\left(y - \frac{b}{4}\right) + e = 0 \qquad \text{(A.1)}$$

that is

$$P(y) = y^4 + \left(c - \frac{3b^2}{8}\right)y^2 + \left(d - \frac{bc}{2} + \frac{b^3}{8}\right)y - \frac{3b^4}{256} + \frac{cb^2}{16} - \frac{db}{4} + e = 0 \qquad \text{(A.2)}$$

Notice that the coefficient of the term of first degree in y is identically equal to zero since

$$d + \frac{b^3}{8} - \frac{bc}{2} = \rho^2(\delta + v) + \rho\left(\delta^2 + v^2 + \Sigma\right) - \rho^3 + \rho^3 - \rho^2(\delta + v) - \rho\left(\delta^2 + v^2 + K\right) = 0 \quad \text{(A.3)}$$

so that we can rewrite $P(y)$ as

$$P(y) = y^4 + \theta y^2 + \phi = 0 \qquad \text{(A.4)}$$

where $\theta = c - \frac{3b^2}{8}$ and $\phi = -\frac{3b^4}{256} + \frac{cb^2}{16} - \frac{db}{4} + e$. Making the change of variable $y^2 = z$, we get

$$\Omega(z) = z^2 + \theta z + \phi = 0 \qquad \text{(A.5)}$$

Notice that θ is always negative since:

$$\begin{aligned}
\theta &= c - \frac{3b^2}{8} = \rho^2 - \rho(\delta + v) - \left(\delta^2 + v^2 + \Sigma\right) - \frac{3}{8}4\rho^2 \\
&= \left(1 - \frac{3}{2}\right)\rho^2 - \rho(\delta + v) - \left(\delta^2 + v^2 + \Sigma\right) \\
&= -\left[\frac{1}{2}\rho^2 + \rho(\delta + v) + \left(\delta^2 + v^2 + \Sigma\right)\right] < 0
\end{aligned}$$

while τ is positive:

$$\phi = -\frac{3b^4}{256} + \frac{cb^2}{16} - \frac{db}{4} + e > 0$$

Indeed, e is greater than zero and so are the first three addends of τ:

$$\begin{aligned}
& -\frac{3b^4}{256} + \frac{cb^2}{16} - \frac{db}{4} \\
= & -\frac{3 \cdot 16}{256}\rho^4 + \frac{\rho^2}{4}\left[\rho^2 - \rho(\delta + v) - \left(\delta^2 + v^2 + \Sigma\right)\right] + \frac{\rho}{2}\left[\rho^2(\delta + v) + \rho\left(\delta^2 + v^2 + \Sigma\right)\right] \\
= & \left(\frac{1}{4} - \frac{3}{16}\right)\rho^4 + \frac{1}{4}\rho^3(\delta + v) + \frac{1}{4}\rho^2\left(\delta^2 + v^2 + \Sigma\right) \\
= & \frac{1}{4}\rho^2\left[\frac{1}{4}\rho^2 + \rho(\delta + v) + \left(\delta^2 + v^2 + \Sigma\right)\right] > 0
\end{aligned}$$

Appendix A.1 The solution of the quartic equation. The resolvent cubic

To study the nature of the solution of the quartic equation, it is now convenient introduce the Decartes–Euler resolvent cubic of $P(y)$. This has the form

$$z^3 + 2\theta z^2 + \left(\theta^2 - 4\phi\right) z = 0 \tag{A.6}$$

which can be written

$$\left(z^2 + 2\theta z + \left(\theta^2 - 4\phi\right)\right) z = 0 \tag{A.7}$$

One solution of the resolvent cubic is of course 0. The other two solutions are

$$z_{1,2} = -\theta \pm \sqrt{\theta^2 - \left(\theta^2 - 4\phi\right)} = -\theta \pm 2\sqrt{\tau} \tag{A.8}$$

Notice that, while the first solution

$$z_1 = -\theta + 2\sqrt{\phi} \tag{A.9}$$

is surely positive (since $\theta < 0$ and $\tau > 0$), the other solution

$$z_2 = -\theta - 2\sqrt{\phi} \tag{A.10}$$

can be positive or negative according to $-\theta \gtrless 2\sqrt{\phi}$. In any case, $z_1 - z_2 = 4\sqrt{\phi} > 0$. That is, z_1 is always greater than z_2.

These two solutions can now be used in the following way to solve the original quartic equation

$$\begin{aligned} x_1 &= \frac{1}{2}\left(\sqrt{z_1} + \sqrt{z_2}\right) - \frac{b}{4};\ x_3 = \frac{1}{2}\left(-\sqrt{z_1} + \sqrt{z_2}\right) - \frac{b}{4} \\ x_2 &= \frac{1}{2}\left(\sqrt{z_1} - \sqrt{z_2}\right) - \frac{b}{4};\ x_4 = \frac{1}{2}\left(-\sqrt{z_1} - \sqrt{z_2}\right) - \frac{b}{4} \end{aligned} \tag{A.11}$$

We can distinguish two cases.

1. z_1 and z_2 are both positive, i.e. $-\theta > 2\sqrt{\phi}$. In this case, x_1 and x_2 are both positive – since $z_1 > z_2$ and $-\frac{b}{4} = \frac{2\rho}{4} = \frac{\rho}{2} > 0$ – while x_3 and x_4 are both negative.[3]

[3] This requires a bit of algebra.
If we show that x_4 is negative, x_3 must also be negative. This is because, by the fundamental theorem of algebra, the product of the four roots must be equal to e, which is positive.
To show that x_4 is negative, rewrite it as

$$\begin{aligned} x_4 &= \frac{1}{2}\left(-\sqrt{z_1} - \sqrt{z_2}\right) - \frac{b}{4} \\ &= -\frac{1}{2}\left[\sqrt{z_1} + \sqrt{z_2} + \frac{b}{2}\right] \end{aligned}$$

2. z_1 is positive while z_2 is negative, i.e. $\theta < 2\sqrt{\tau}$. In this case the solutions of the quartic equation are two pairs of complex conjugate roots. They are

$$\begin{aligned} x_{1,2} &= \frac{1}{2}\sqrt{z_1} - \frac{b}{4} \pm \sqrt{z_2} i \\ x_{3,4} &= -\frac{1}{2}\sqrt{z_1} - \frac{b}{4} \pm \sqrt{z_2} i \end{aligned}$$

Appendix A.2 The solution of the quartic equation. The biquadratic equation

The quartic equation also can be reduced to a biquadratic equation by a double change of variable: first, from x to $y = x + \frac{b}{4}$, and then from y to $z = y^2$.

Now, to show that x_4 is negative, it is enough to show that

$$\sqrt{z_1} + \frac{b}{2} = \sqrt{z_1} - \rho > 0$$

since $\sqrt{z_2}$ is positive.

Suppose for analytical convenience that $K = e = 0$, so that $-\theta$

$$-\theta = \frac{1}{2}\rho^2 + \rho(\delta + v) + \left(\delta^2 + v^2 + K\right)$$

becomes

$$\frac{1}{2}\rho^2 + \rho(\delta + v) + \left(\delta^2 + v^2\right)$$

while τ

$$\tau = \frac{1}{4}\rho^2 \left[\frac{1}{4}\rho^2 + \rho(\delta + v) + \left(\delta^2 + v^2 + K\right)\right] + e$$

becomes

$$\frac{1}{4}\rho^2 \left[\frac{1}{4}\rho^2 + \rho(\delta + v) + \left(\delta^2 + v^2\right)\right]$$

Notice that this assumption reduces $-\theta$ and τ, and thus $\sqrt{z_1} = \sqrt{-\theta + \sqrt{\tau}}$.

Then, $\sqrt{z_1} + \frac{b}{2}$ becomes:

$$\begin{aligned} & \sqrt{\frac{1}{2}\rho^2 + \rho(\delta + v) + \left(\delta^2 + v^2\right) + 2\sqrt{\frac{1}{4}\rho^2 \left[\frac{1}{4}\rho^2 + \rho(\delta + v) + \left(\delta^2 + v^2\right)\right]}} - \rho \\ = & \sqrt{\frac{1}{2}\rho^2 + \rho(\delta + v) + \left(\delta^2 + v^2\right) + \rho\left(\frac{1}{2}\rho + (\delta + v)\right)} - \rho \\ = & \sqrt{\rho^2 + 2\rho(\delta + v) + \left(\delta^2 + v^2\right)} - \rho \\ = & (\rho + \delta + v) - \rho = \delta + v > 0 \end{aligned}$$

Thus, we have shown that x_4 is negative.

The first change produces the following equation:

$$P(y) = \left(y - \frac{b}{4}\right)^4 + b\left(y - \frac{b}{4}\right)^3 + c\left(y - \frac{b}{4}\right)^2 + d\left(y - \frac{b}{4}\right) + e = 0$$

that is

$$P(y) = y^4 + \left(c - \frac{3b^2}{8}\right)y^2 + \left(d - \frac{bc}{2} + \frac{b^3}{8}\right)y - \frac{3b^4}{256} + \frac{cb^2}{16} - \frac{db}{4} + e = 0$$

Again, notice that the coefficient of the term of first degree in y is identically equal to zero since

$$\begin{aligned} d + \frac{b^3}{8} - \frac{bc}{2} &= \rho^2(\delta + v) + \rho\left(\delta^2 + v^2 + L - \frac{N}{M^*}\right) - \rho^3 + \rho^3 - \rho^2(\delta + v) + \\ -\rho\left(\delta^2 + v^2 + L - \frac{N}{M^*}\right) &= 0 \end{aligned}$$

so that we can rewrite $P(y)$ as

$$P(y) = y^4 + \theta y^2 + \tau = 0$$

where $\theta = c - \frac{3b^2}{8}$ and $\tau = -\frac{3b^4}{256} + \frac{cb^2}{16} - \frac{db}{4} + e$. Making the change of variable $y^2 = z$, we get

$$\Omega(z) = z^2 + \theta z + \tau = 0 \tag{A.12}$$

Notice that θ is always negative since:

$$\begin{aligned} \theta &= c - \frac{3b^2}{8} = \rho^2 - \rho(\delta + v) - \left(\delta^2 + v^2 + L - \frac{N}{M^*}\right) - \frac{3}{8}4\rho^2 \\ &= \left(1 - \frac{3}{2}\right)\rho^2 - \rho(\delta + v) - \left(\delta^2 + v^2 + L - \frac{N}{M^*}\right) \\ &= -\left[\frac{1}{2}\rho^2 + \rho(\delta + v) + \left(\delta^2 + v^2 + L - \frac{N}{M^*}\right)\right] < 0 \end{aligned}$$

since $N < 0$. Exactly as above, τ is positive:

$$\tau = -\frac{3b^4}{256} + \frac{cb^2}{16} - \frac{db}{4} + e > 0$$

Indeed, e is greater than zero and so are the first three addends of τ:

$$
\begin{aligned}
& -\frac{3b^4}{256}+\frac{cb^2}{16}-\frac{db}{4} \\
= & -\frac{3\cdot 16}{256}\rho^4+\frac{\rho^2}{4}\left[\rho^2-\rho\left(\delta+v\right)-\left(\delta^2+v^2+L-\frac{N}{M^*}\right)\right]+\frac{\rho}{2}\left[\rho^2\left(\delta+v\right)+\rho\left(\delta^2+v^2+L-\frac{N}{M^*}\right)\right. \\
= & \left(\frac{1}{4}-\frac{3}{16}\right)\rho^4+\frac{1}{4}\rho^3\left(\delta+v\right)+\frac{1}{4}\rho^2\left(\delta^2+v^2+L-\frac{N}{M^*}\right) \\
= & \frac{1}{4}\rho^2\left[\frac{1}{4}\rho^2+\rho\left(\delta+v\right)+\left(\delta^2+v^2+L-\frac{N}{M^*}\right)\right] \\
= & \frac{1}{16}\rho^4+\frac{1}{4}\rho^3\left(\delta+v\right)+\frac{1}{4}\rho^2\left(\delta^2+v^2+L-\frac{N}{M^*}\right)>0
\end{aligned}
$$

Appendix A.3 Solving the biquadratic equation

Instead of using the cubic resolvent, we now proceed by directly solving the biquadratic equation. Begin with the quadratic equation (A.12). Its solutions are given by:

$$z_{1,2}=\frac{-\theta\pm\sqrt{\theta^2-4\tau}}{2}=\frac{-\theta\pm\sqrt{\left(\theta-2\sqrt{\tau}\right)\left(\theta+2\sqrt{\tau}\right)}}{2}$$

Since θ is negative, the radicand will be positive, and the roots real, if and only if $\theta+2\sqrt{\tau}$ is negative, i.e. if and only if $\theta<-2\sqrt{\tau}$ or equivalently $-\theta>2\sqrt{\tau}$, which is the same condition found above. In such a case, both z_1 and z_2 are positive. The corresponding negative roots for x are:

$$
\begin{aligned}
x_1 &= -\sqrt{\frac{-\theta+\sqrt{\theta^2-4\tau}}{2}}-\frac{b}{4}=-\sqrt{\frac{-\theta+\sqrt{\theta^2-4\tau}}{2}}+\frac{\rho}{2} \\
x_2 &= -\sqrt{\frac{-\theta-\sqrt{\theta^2-4\tau}}{2}}-\frac{b}{4}=-\sqrt{\frac{-\theta-\sqrt{\theta^2-4\tau}}{2}}+\frac{\rho}{2}
\end{aligned}
$$

while the positive ones are:

$$
\begin{aligned}
x_3 &= \sqrt{\frac{-\theta+\sqrt{\theta^2-4\tau}}{2}}-\frac{b}{4}=\sqrt{\frac{-\theta+\sqrt{\theta^2-4\tau}}{2}}+\frac{\rho}{2} \\
x_4 &= \sqrt{\frac{-\theta-\sqrt{\theta^2-4\tau}}{2}}-\frac{b}{4}=\sqrt{\frac{-\theta-\sqrt{\theta^2-4\tau}}{2}}+\frac{\rho}{2}
\end{aligned}
$$

If instead $\theta>-2\sqrt{\tau}$, z_1 and z_2 are complex conjugates. We know that in this case the x solutions are:

$$
\begin{aligned}
x_{1,2} &= -\frac{1}{2}\sqrt{z_1} - \frac{b}{4} \pm \sqrt{z_2}i; \\
x_{3,4} &= \frac{1}{2}\sqrt{z_1} - \frac{b}{4} \pm \sqrt{z_2}i.
\end{aligned}
$$

In any case to these four solutions, be they real or complex, we must add the fifth solution $x_5 = \rho + \delta$.

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