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"GENERALIZED FUZZY DIFFERENTIABILITY WITH LU-PARAMETRIC REPRESENTATION"

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Generalized Fuzzy Differentiability with LU-parametric Representation

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Abstract

In the present paper, we use a new generalization of the Hukuhara difference and derivative for fuzzy-valued functions, and we study several properties of the new concepts in the setting of the LU-parametric representation of fuzzy numbers, assessed both from theoretical and computational points of view.

Key words: Fuzzy-valued function, Generalized Hukuhara differentiability, Generalized fuzzy derivative, LU-parametric fuzzy number

1 Introduction

The interest for differentiability of fuzzy-valued functions is increasing in the recent literature; several generalized fuzzy derivative concepts are studied in relation with the similar notions in [2], [27], [28].

These new generalized derivatives are motivated by their usefulness in a very quickly developing area at the intersection of set-valued analysis and fuzzy sets, namely, the area of fuzzy analysis and fuzzy differential equations, e.g., [1], [3], [10], [11], [12], [13], [17], [19], [20], [22], [31].

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The g-difference proposed in [25], [24], [26] has a great advantage over peer concepts (e.g. [6]), namely that it always exists, and relatively simple expressions allow its calculation.

In [4], based on the fuzzy gH-difference and g-difference, new generalizations of the differentiability for fuzzy-number-valued functions are defined and analyzed and connections to the ideas of [14], [15], [16] are shown; in particular, a new, very general fuzzy differentiability concept is defined and studied, the so-called g-derivative, and it is shown that the g-derivative is the most general among all similar definitions. The properties we obtained in [4], show characterization of the new g-differentiability, an interesting minimality property and some computational results.

In this paper, the LU-parametric representation proposed in [8], [29], [30] is considered together with the new derivative concepts and we investigate numerical differentiation algorithms.

The paper is organized as follows; section 2 introduces the fuzzy g-difference and the g-derivative; in section 3, the LU-parametric representation of fuzzy numbers and computational procedures are presented; the paper concludes with section 4, where some examples and computational tools are presented.

2 Generalized fuzzy difference and generalized fuzzy derivative

One of the first definitions of difference and derivative for set-valued functions was given by Hukuhara [9] (H-difference and H-derivative); it has been extended to the fuzzy case in [21] and applied to fuzzy differential equations (FDE) by many authors in several papers (see [6], [10], [11], [12], [19]). But the H-derivative in FDE suffers certain disadvantages (see [2], [24]) related to the fact that Minkowski addition does not possess an inverse subtraction. On the other hand, a more general definition of subtraction for compact convex sets, and in particular for compact intervals, has been introduced by several authors. Markov [14], [15] defined a non-standard difference, also called *inner*-difference, and extended its use to interval arithmetic and to interval calculus. In the setting of Hukuhara difference, the interval and fuzzy generalized Hukuhara differences have been recently examined in [25], [26].

We start with a brief account of these concepts.

Let $\mathcal{K}_C^n$ be the space of nonempty compact and convex sets of $\mathbb{R}^n$. The Hukuhara H-difference has been introduced as a set C for which $A \ominus_H B = C \iff A = B + C$ and an important property of $\ominus_H$ is that $A \ominus_H A = \{0\}$ $\forall A \in \mathcal{K}_C^n$ and $(A + B) \ominus_H B = A$, $\forall A, B \in \mathcal{K}_C^n$. The H-difference is unique, but it does not always exist (a necessary condition for $A \ominus_H B$ to exist is that A contains a translate $\{c\} + B$ of B). A generalization of the Hukuhara difference aims to overcome this situation. The generalized Hukuhara difference of two sets $A, B \in \mathcal{K}^n_C$ (gH-difference for short) is defined as follows

$$A \ominus_{gH} B = C \iff \begin{cases} (a)\ A = B + C \\ \text{or } (b)\ B = A + (-1)C \end{cases} \quad (1)$$

We will denote $\mathbb{R}_{\mathcal{F}}$ the set of fuzzy numbers, i.e. normal, fuzzy convex, upper semicontinuous and compactly supported fuzzy sets defined over the real line. Fundamental concepts in fuzzy theory are the *support*, the *level-sets* (or *level-cuts*) and the *core* of a fuzzy number.

Here, $cl(X)$ denotes the closure of set X.

Definition 1. Let $u \in \mathbb{R}_{\mathcal{F}}$ be a fuzzy number. For $\alpha \in]0,1]$, the α-level set of u (or simply the $\alpha-$cut) is defined by $[u]_\alpha = \{x|x \in \mathbb{R}, u(x) \geq \alpha\}$ and for $\alpha = 0$ by the closure of the support $[u]_0 = cl\{x|x \in \mathbb{R}, u(x) > 0\}$. The *core* of u is the set of elements of $\mathbb{R}$ having membership grade 1, i.e., $[u]_1 = \{x|x \in \mathbb{R}, u(x) = 1\}$.

It is well-known that the $level-cuts$ are "nested", i.e. $[u]_\alpha \subseteq [u]_\beta$ for $\alpha > \beta$. A fuzzy set u is a fuzzy number if and only if the $\alpha - cuts$ are nonempty compact intervals of the form $[u]_\alpha = [u^-_\alpha, u^+_\alpha] \subset \mathbb{R}$. The "nested" property is the basis for the LU representation (L for lower, U for upper) (see [7], [29], [30]).

Proposition 2. *A fuzzy number u is completely determined by any pair $u = (u^-, u^+)$ of functions $u^-, u^+ : [0,1] \longrightarrow \mathbb{R}$, defining the end-points of the $\alpha - cuts$, satisfying the three conditions:*
(i) $u^- : \alpha \longrightarrow u^-_\alpha \in \mathbb{R}$ is a bounded monotonic nondecreasing left-continuous function $\forall \alpha \in]0,1]$ and right-continuous for $\alpha = 0$;
(ii) $u^+ : \alpha \longrightarrow u^+_\alpha \in \mathbb{R}$ is a bounded monotonic nonincreasing left-continuous function $\forall \alpha \in]0,1]$ and right-continuous for $\alpha = 0$;
(iii) $u^-_1 \leq u^+_1$ for $\alpha = 1$, which implies $u^-_\alpha \leq u^+_\alpha$ $\forall \alpha \in [0,1]$.

The following result is well known [18]:

Proposition 3. *Let $\{U_\alpha | \alpha \in]0,1]\}$ be a family of real intervals such that the following three conditions are satisfied:*
1. U_α is a nonempty compact interval for all $\alpha \in]0,1]$;
2. if $0 < \alpha < \beta \leq 1$ then $U_\beta \subseteq U_\alpha$;
3. given any nondecreasing sequence $\alpha_n \in]0,1]$ with $\lim\limits_{n \longrightarrow \infty} \alpha_n = \alpha > 0$ we have

$U_\alpha = \bigcap_{n=1}^{\infty} U_{\alpha_n}$.
Then there exists a unique LU-fuzzy quantity u *such that* $[u]_\alpha = U_\alpha$ *for all* $\alpha \in]0,1]$ *and* $[u]_0 = cl\left(\bigcup_{\alpha\in]0,1]} U_\alpha\right)$.

We refer to the functions $u^-_{(.)}$ and $u^+_{(.)}$ as the lower and upper branches of u, respectively. A trapezoidal fuzzy interval, denoted by $u = \langle a,b,c,d\rangle$, where $a \leq b \leq c \leq d$, has $\alpha - cuts$ $[u]_\alpha = [a+\alpha(b-a), d-\alpha(d-c)]$, $\alpha \in [0,1]$, obtaining a triangular fuzzy number if $b = c$.

The addition $u+v$ and the scalar multiplication ku are defined as having the level cuts

$$[u+v]_\alpha = [u]_\alpha + [v]_\alpha = \{x+y | x \in [u]_\alpha,\ y \in [v]_\alpha\}$$
$$[ku]_\alpha = k[u]_\alpha = \{kx | x \in [u]_\alpha\},\ [0]_\alpha = \{0\}\ \forall \alpha \in [0,1]$$

The subtraction of fuzzy numbers $u - v$ is defined as the addition $u + (-v)$ where $-v = (-1)v$, but in general $u - u \neq 0$.

The standard Hukuhara difference (H-difference $\ominus_H$) is defined by $u \ominus_H v = w \iff u = v + w$, being $+$ the standard fuzzy addition; if $u \ominus_H v$ exists, its $\alpha - cuts$ are $[u \ominus_H v]_\alpha = [u^-_\alpha - v^-_\alpha, u^+_\alpha - v^+_\alpha]$. It is well known that $u \ominus_H u = 0$.

The Hausdorff distance on $\mathbb{R}_\mathcal{F}$ is defined by

$$D(u,v) = \sup_{\alpha\in[0,1]} \left\{ \| [u]_\alpha \ominus_{gH} [v]_\alpha \|_* \right\},$$

where, for an interval $[a,b]$, the norm is

$$\|[a,b]\|_* = \max\{|a|, |b|\}$$

and it is well known that $(\mathbb{R}_\mathcal{F}, D)$ is a complete metric space.

Definition 4. Given two fuzzy numbers $u, v \in \mathbb{R}_\mathcal{F}$, the generalized Hukuhara difference (gH-difference for short) is the fuzzy number w, if it exists, such that

$$u \ominus_{gH} v = w \iff \begin{cases} \quad\ (i) \quad u = v + w \\ or \quad (ii) \quad v = u - w \end{cases}. \tag{2}$$

It is easy to show that *(i)* and *(ii)* are both valid if and only if w is a crisp number.

The following properties were obtained in [25].

Proposition 5. *([25]) Let* $u, v \in \mathbb{R}_\mathcal{F}$ *be two fuzzy numbers; then*
i) if the gH-difference exists, it is unique;

ii) $u \ominus_{gH} v = u \ominus_H v$ *or* $u \ominus_{gH} v = -(v \ominus_H u)$ *whenever the expressions on the right exist; in particular,* $u \ominus_{gH} u = u \ominus_H u = 0$,
iii) if $u \ominus_{gH} v$ *exists in the sense* (i), *then* $v \ominus_{gH} u$ *exists in the sense* (ii) *and viceversa,*
iv) $(u + v) \ominus_{gH} v = u$,
v) $0 \ominus_{gH} (u \ominus_{gH} v) = v \ominus_{gH} u$,
vi) $u \ominus_{gH} v = v \ominus_{gH} u = w$ *if and only if* $w = -w$; *furthermore,* $w = 0$ *if and only if* $u = v$.

In [25], [26] a new difference between fuzzy numbers was proposed, a difference that always exists.

Definition 6. The generalized difference (g-difference for short) of two fuzzy numbers $u, v \in \mathbb{R}_{\mathcal{F}}$ is given by its level sets as

$$[u \ominus_g v]_\alpha = cl \bigcup_{\beta \geq \alpha} ([u]_\beta \ominus_{gH} [v]_\beta), \forall \alpha \in [0, 1], \tag{3}$$

where the gH-difference $\ominus_{gH}$ is with interval operands $[u]_\beta$ and $[v]_\beta$.

The following propositions give simplified notation for $u \ominus_g v$ (see [4]) and some properties.

Proposition 7. *For any two fuzzy numbers* $u, v \in \mathbb{R}_{\mathcal{F}}$ *the two g-differences* $u \ominus_g v$ *and* $v \ominus_g u$ *exist and, for any* $\alpha \in [0, 1]$, *we have* $u \ominus_g v = -(v \ominus_g u)$ *with*

$$\begin{gathered} [u \ominus_g v]_\alpha = [d_\alpha^-, d_\alpha^+] \text{ and } [v \ominus_g u]_\alpha = [-d_\alpha^+, -d_\alpha^-] \\ \text{where} \\ d_\alpha^- = \inf(D_\alpha), \quad d_\alpha^+ = \sup(D_\alpha) \\ \text{and the sets } D_\alpha \text{ are} \\ D_\alpha = \{u_\beta^- - v_\beta^- | \beta \geq \alpha\} \cup \{u_\beta^+ - v_\beta^+ | \beta \geq \alpha\}. \end{gathered} \tag{4}$$

Let us consider the fuzzy inclusion defined as $u \subseteq v \iff u(x) \leq v(x), \forall x \in \mathbb{R} \iff [u]_\alpha \subseteq [v]_\alpha, \forall \alpha \in [0, 1]$. The following proposition provides a minimality property for the g-difference.

Proposition 8. *The g-difference* $u \ominus_g v$ *is the smallest fuzzy number* w *in the sense of fuzzy inclusion such that*

$$u \subseteq v + w \text{ and } v \subseteq u - w;$$

Proposition 9. *Let* $u, v \in \mathbb{R}_{\mathcal{F}}$ *be two fuzzy numbers; then*
i) $u \ominus_g v = u \ominus_{gH} v$, *whenever the expression on the right exists; in particular* $u \ominus_g u = 0$,
ii) $(u + v) \ominus_g v = u$,

iii) $0 \ominus_g (u \ominus_g v) = v \ominus_g u$,
iv) $u \ominus_g v = v \ominus_g u = w$ *if and only if* $w = -w$*; furthermore,* $w = 0$ *if and only if* $u = v$.

The connection between the gH-difference, the g-difference and the Hausdorff distance adds a geometric interpretation for the differences constructed.

Proposition 10. *We have*

$$D(u, v) = \|u \ominus_g v\|$$

where $\|\cdot\| = D(\cdot, 0)$.

Generalized differentiability concepts were first considered for interval-valued functions in the works of Markov ([14], [16]). This line of research is continued by several papers [2], [5], [27] etc. dealing with interval and fuzzy-valued functions.

Definition 11. *Let* $x_0 \in]a, b[$ *and* h *be such that* $x_0 + h \in]a, b[$*, then the gH-derivative of a function* $f :]a, b[\to \mathbb{R}_{\mathcal{F}}$ *at* x_0 *is defined as*

$$f'_{gH}(x_0) = \lim_{h \to 0} \frac{1}{h} [f(x_0 + h) \ominus_{gH} f(x_0)]. \tag{5}$$

If $f'_{gH}(x_0) \in \mathbb{R}_{\mathcal{F}}$ *satisfying (5) exists, we say that* f *is generalized Hukuhara differentiable (gH-differentiable for short) at* x_0.

As we have seen in equation (3), both fuzzy gH-difference and g-difference are based on the interval gH-difference for each α-cut of the involved fuzzy numbers; this level characterization is obviously inherited by the fuzzy gH-derivative, with respect to the level-wise gH-derivative.

Definition 12. *Let* $x_0 \in]a, b[$ *and* h *be such that* $x_0 + h \in]a, b[$*, then the level-wise gH-derivative (LgH-derivative for short) of a function* $f :]a, b[\to \mathbb{R}_{\mathcal{F}}$ *at* x_0 *is defined as the set of interval-valued gH-derivatives, if they exist,*

$$f'_{LgH}(x_0)_\alpha = \lim_{h \to 0} \frac{1}{h} \left([f(x_0 + h)]_\alpha \ominus_{gH} [f(x_0)]_\alpha \right). \tag{6}$$

If $f'_{LgH}(x_0)_\alpha$ *is a compact interval for all* $\alpha \in [0, 1]$*, we say that* f *is level-wise generalized Hukuhara differentiable (LgH-differentiable for short) at* x_0 *and the family of intervals* $\{f'_{LgH}(x_0)_\alpha | \alpha \in [0, 1]\}$ *is the* LgH-derivative *of* f *at* x_0*, denoted by* $f'_{LgH}(x_0)$.

Theorem 13. *Let* $f :]a, b[\to \mathbb{R}_{\mathcal{F}}$ *be such that* $[f(x)]_\alpha = [f^-_\alpha(x), f^+_\alpha(x)]$*. Suppose that the functions* $f^-_\alpha(x)$ *and* $f^+_\alpha(x)$ *are real-valued functions, differentiable w.r.t.* x*, uniformly w.r.t.* $\alpha \in [0, 1]$*. Then the function* $f(x)$ *is gH-differentiable at a fixed* $x \in]a, b[$ *if and only if one of the following two cases*

holds:
a) $(f_\alpha^-)'(x)$ *is increasing,* $(f_\alpha^+)'(x)$ *is decreasing as functions of* α, *and* $\left(f_1^-\right)'(x) \leq \left(f_1^+\right)'(x)$, *or*
b) $(f_\alpha^-)'(x)$ *is decreasing,* $(f_\alpha^+)'(x)$ *is increasing as functions of* α, *and* $\left(f_1^+\right)'(x) \leq \left(f_1^-\right)'(x)$.
Also, $\forall\alpha \in [0,1]$ *we have*

$$\left[f'_{gH}(x)\right]_\alpha = [\min\{\left(f_\alpha^-\right)'(x), \left(f_\alpha^+\right)'(x)\}, \max\{\left(f_\alpha^-\right)'(x), \left(f_\alpha^+\right)'(x)\}] \tag{7}$$

According to Theorem 13, for the definition of gH-differentiability when $f_\alpha^-(x)$ and $f_\alpha^+(x)$ are both differentiable, we distinguish two cases, corresponding to (i) and (ii) of (2).

Definition 14. *Let* $f : [a,b] \longrightarrow \mathbb{R}_\mathcal{F}$ *and* $x_0 \in]a,b[$ *with* $f_\alpha^-(x)$ *and* $f_\alpha^+(x)$ *both differentiable at* x_0. *We say that*
- *f is (i)-gH-differentiable at* x_0 *if*

$$\text{(i.)} \qquad \left[f'_{gH}(x_0)\right]_\alpha = [\left(f_\alpha^-\right)'(x_0), \left(f_\alpha^+\right)'(x_0)], \forall\alpha \in [0,1] \tag{8}$$

- *f is (ii)-gH-differentiable at* x_0 *if*

$$\text{(ii.)} \qquad \left[f'_{gH}(x_0)\right]_\alpha = [\left(f_\alpha^+\right)'(x_0), \left(f_\alpha^-\right)'(x_0)], \forall\alpha \in [0,1]. \tag{9}$$

Definition 15. *Let* $x_0 \in]a,b[$ *and h be such that* $x_0 + h \in]a,b[$, *then the g-derivative of a function* $f :]a,b[\to \mathbb{R}_\mathcal{F}$ *at* x_0 *is defined as*

$$f'_g(x_0) = \lim_{h\to 0} \frac{1}{h}[f(x_0+h) \ominus_g f(x_0)]. \tag{10}$$

If $f'_g(x_0) \in \mathbb{R}_\mathcal{F}$ *satisfying (10) exists, we say that f is generalized differentiable (g-differentiable for short) at* x_0.

The next result provides a first expression for the g-derivative and its connection to the interval gH-derivative of the level sets. According to the result that the existence of the gH-differences for all level sets is sufficient to define the g-difference, the uniform LgH-differentiability is sufficient for the g-differentiability.

Theorem 16. *([4]) Let* $f :]a,b[\to \mathbb{R}_\mathcal{F}$ *be uniformly LgH-differentiable at* x_0. *Then f is g-differentiable at* x_0 *and, for any* $\alpha \in [0,1]$,

$$[f'_g(x_0)]_\alpha = cl\left(\bigcup_{\beta\geq\alpha} f'_{LgH}(x_0)_\beta\right)$$

In the following Theorem we give a characterization and a practical formula for the g-derivative.

Theorem 17. *([4]) Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ be such that $[f(x)]_\alpha = [f_\alpha^-(x), f_\alpha^+(x)]$. If $f_\alpha^-(x)$ and $f_\alpha^+(x)$ are differentiable real-valued functions with respect to x, uniformly for $\alpha \in [0,1]$, then $f(x)$ is g-differentiable and we have*

$$\left[f_g'(x)\right]_\alpha = \left[\inf_{\beta \geq \alpha} \min\{\left(f_\beta^-\right)'(x), \left(f_\beta^+\right)'(x)\}, \sup_{\beta \geq \alpha} \max\{\left(f_\beta^-\right)'(x), \left(f_\beta^+\right)'(x)\}\right]. \tag{11}$$

The next Theorem shows a minimality property for the g-derivative ([4]).

Theorem 18. *Let f be uniformly LgH-differentiable. Then $f_g'(x)$, for a fixed x, is the smallest fuzzy number $w \in \mathbb{R}_{\mathcal{F}}$ (in the sense of fuzzy inclusion) such that $f_{LgH}'(x)_\alpha \subseteq [w]_\alpha$ for all $\alpha \in [0,1]$.*

We will assume, for the rest of this section, that $f_\alpha^-(x)$ and $f_\alpha^+(x)$ are differentiable w.r.t. x for all α.

Definition 19. *We say that a point $x \in]a,b[$ is an l-critical point of f if it is a critical point for the length function $len([f(x)]_\alpha) = f_\alpha^+(x) - f_\alpha^-(x)$ for some $\alpha \in [0,1]$.*

If f is gH-differentiable everywhere in its domain the switch at every level should happen at the same time, i.e., $\frac{d}{dx} len([f(x)]_\alpha) = (f_\alpha^+(x) - f_\alpha^-(x))' = 0$ at the same point x for all $\alpha \in [0,1]$ ([4]).

Definition 20. *We say that a point $x_0 \in]a,b[$ is a switching point for the gH-differentiability of f, if in any neighborhood V of x_0 there exist points $x_1 < x_0 < x_2$ such that*
type-I switch point) at x_1 (8) holds while (9) does not hold and at x_2 (9) holds and (8) does not hold, or
type-II switch point) at x_1 (9) holds while (8) does not hold and at x_2 (8) holds and (9) does not hold.

Obviously, any switching point is also an *l-critical* point. Indeed, if x_0 is a switching point then $[(f_\alpha^-)'(x_0), (f_\alpha^+)'(x_0)] = [(f_\alpha^+)'(x_0), (f_\alpha^-)'(x_0)]$ and so $\left(f_0^+\right)'(x_0) = \left(f_0^-\right)'(x_0)$ and $len(f(x_0))' = 0$. Clearly, not all *l-critical* points are also switching points.

Definition 21. ([4]) *We say that an interval $S = [x_1, x_2] \subseteq]a,b[$, where f is g-differentiable, is a transitional region for the differentiability of f, if in any neighborhood $(x_1 - \delta, x_2 + \delta) \supset S$, $\delta > 0$, there exist points $x_1 - \delta < \xi_1 < x_1$ and $x_2 < \xi_2 < x_2 + \delta$ such that*
type-I switch region) at ξ_1 (8) holds while (9) does not hold and at ξ_2 (9) holds

and (8) does not hold, or
type-II switch region) at ξ_1 (9) holds while (8) does not hold and at ξ_2 (8) holds and (9) does not hold.

3 Generalized differentiability with LU-parametric fuzzy numbers

The Lower-Upper (LU) representation of a fuzzy number is a result based on the well known Negoita-Ralescu representation theorem, stating essentially that the membership form and the α-cut form of a fuzzy number u are equivalent and in particular, the α-cuts $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$ uniquely represent u, provided that the two functions $\alpha \longrightarrow u_\alpha^-$ and $\alpha \longrightarrow u_\alpha^+$, w.r.t. α, are left continuous for all $\alpha \in]0,1]$, right continuous for $\alpha = 0$, monotonic (u_α^- increasing, u_α^+ decreasing) and $u_1^- \leq u_1^+$ (for $\alpha = 1$).

On the other hand, it is well known that monotonic functions have at most a countable number of points of discontinuity and a countable number of points where the derivative does not exist.

Denote the corresponding points by the strictly increasing sequence $(\alpha_j)_{j \in J}$ with $0 < \alpha_j < \alpha_{j+1} < 1$ and $J = \emptyset$ (empty set) or $J = \{1, 2..., p\}$ (finite set) or $J = \mathbb{N}$ (set of natural numbers).

Then the two functions u_α^-, u_α^+ are differentiable internally to each of the subintervals $[\alpha_{j-1}, \alpha_j]$ i.e., they are formed by a family of differentiable monotonic "pieces", and their restrictions to each subinterval are monotonic and differentiable.

The LU-parametric representation of fuzzy numbers, proposed in [8], [29], is shown to have a great application potential. The lower and upper functions $\alpha \longrightarrow u_\alpha^-$ and $\alpha \longrightarrow u_\alpha^+$ of a fuzzy number $u \in \mathbb{R}_\mathcal{F}$ can be expressed in LU-parametric form as follows.

First, choose a family of "standardized" differentiable and increasing shape functions $p : [0,1] \longrightarrow [0,1]$, depending on two parameters $\beta_0, \beta_1 \geq 0$, such that
1. $p(0) = 0$, $p(1) = 1$,
2. $p'(0) = \beta_0$, $p'(1) = \beta_1$ and
3. $p(t)$ is increasing on $[0,1]$ if and only if $\beta_0, \beta_1 \geq 0$.

One of the simplest shape functions is, e.g., the (2,2)-rational spline

$$p_{rat}(t; \beta_0, \beta_1) = \frac{t^2 + \beta_0 t(1-t)}{1 + (\beta_0 + \beta_1 - 2)t(1-t)}. \qquad (12)$$

We remark that function $p_{rat}(t;\beta_0,\beta_1)$ is linear if $\beta_0=\beta_1=1$ and is quadratic if $\beta_0+\beta_1=2$, $\beta_0\neq\beta_1$; as we will see, this is an interesting fact, because linear and quadratic shapes are reproduced exactly (without approximation error) by the simpler LU-parametric form.

The shape functions $p(t;\beta_0,\beta_1)$ are adopted to represent the functions $u^-_{(.)}$ and $u^+_{(.)}$ "piecewise" on a decomposition of the interval $[0,1]$ into N subintervals $0=\alpha_0<\alpha_1<...<\alpha_{i-1}<\alpha_i<...<\alpha_N=1$; at the extremal points of each subinterval $I_i=[\alpha_{i-1},\alpha_i]$, the values $u^-_{0,i}$, $u^+_{0,i}$, $u^-_{1,i}$, $u^+_{1,i}$ and the first derivatives (slopes) $d^-_{0,i}$, $d^+_{0,i}$, $d^-_{1,i}$, $d^+_{1,i}$ of the two functions are then assumed to be given, i.e.

$$u^-_{\alpha_{i-1}}=u^-_{0,i}\ \ ,\ \ u^+_{\alpha_{i-1}}=u^+_{0,i}\ \ ,\ \ u^-_{\alpha_i}=u^-_{1,i}\ \ ,\ \ u^+_{\alpha_i}=u^+_{1,i} \tag{13}$$
$$(u^-)'_{\alpha_{i-1}}=d^-_{0,i}\ \ ,\ \ (u^+)'_{\alpha_{i-1}}=d^+_{0,i}\ \ ,\ \ (u^-)'_{\alpha_i}=d^-_{1,i}\ \ ,\ \ (u^+)'_{\alpha_i}=d^+_{1,i} \tag{14}$$

and, for $\alpha\in[\alpha_{i-1},\alpha_i]$ and $i=1,2.,,,N$, we write

$$u^-_\alpha=u^-_{0,i}+(u^-_{1,i}-u^-_{0,i})p^-_i\left(t_\alpha;\beta^-_{0,i},\beta^-_{1,i}\right) \tag{15}$$
$$u^+_\alpha=u^+_{0,i}+(u^+_{1,i}-u^+_{0,i})p^+_i\left(t_\alpha;\beta^+_{0,i},\beta^+_{1,i}\right). \tag{16}$$

where

$$\beta^-_{j,i}=\frac{\alpha_i-\alpha_{i-1}}{u^-_{1,i}-u^-_{0,i}}d^-_{j,i}\text{ and }\beta^+_{j,i}=-\frac{\alpha_i-\alpha_{i-1}}{u^+_{1,i}-u^+_{0,i}}d^+_{j,i}\text{ for }j=0,1$$

and

$$t_\alpha=\frac{\alpha-\alpha_{i-1}}{\alpha_i-\alpha_{i-1}},\quad \alpha\in[\alpha_{i-1},\alpha_i].$$

For simplicity, in this presentation we will assume that $J=\emptyset$ (empty set); otherwise, we can repeat the following results on each of the subintervals.

So, u is assumed to be a fuzzy number with α-cuts $[u]_\alpha=[u^-_\alpha,u^+_\alpha]$ and $\alpha\longrightarrow u^-_\alpha$, $\alpha\longrightarrow u^+_\alpha$ monotonic and differentiable w.r.t. α.

For $\alpha\in[0,1]$, let δu^-_α and δu^+_α denote the first derivatives of u^-_α and u^+_α w.r.t. α (for $\alpha=0$ they are right derivatives, for $\alpha=1$ they are left derivatives).

The following lemma is immediate.

Lemma 22. *Two differentiable functions u^-_α, u^+_α define a fuzzy number if and* *only if for all* $\alpha \in [0,1]$ *we have*

$$\begin{cases} u_1^- \leq u_1^+ \\ \delta u_\alpha^- \geq 0, \forall \alpha \in [0,1] \\ \delta u_\alpha^+ \leq 0, \forall \alpha \in [0,1] \end{cases} OR \begin{cases} u_1^+ \leq u_1^- \\ \delta u_\alpha^- \leq 0, \forall \alpha \in [0,1] \\ \delta u_\alpha^+ \geq 0, \forall \alpha \in [0,1] \end{cases} . \tag{17}$$

A fuzzy number with differentiable lower and upper functions is obtained by taking $u_{1,i}^- = u_{0,i+1}^- =: u_i^-$, $u_{1,i}^+ = u_{0,i+1}^+ =: u_i^+$ and $d_{1,i}^- = d_{0,i+1}^- =: \delta u_i^-$, $d_{1,i}^+ = d_{0,i+1}^+ =: \delta u_i^+$. This requires $4(N+1)$ parameters and we will write (assuming $N \geq 1$)

$$u = (\alpha_i; u_i^-, \delta u_i^-, u_i^+, \delta u_i^+)_{i=0,1,...,N} \text{ with} \tag{18}$$

$$u_0^- \leq u_1^- \leq ... \leq u_N^- \leq u_N^+ \leq u_{N-1}^+ \leq ... \leq u_0^+ \tag{19}$$

$$\delta u_i^- \geq 0,\ \delta u_i^+ \leq 0,\ i = 0,1,...,N. \tag{20}$$

The functions u_α^- and u_α^+ are then computed according to (15)-(16).

In general, as illustrated in [8], [29], the fact that the slopes are available reduces greatly the number of points α_i needed to reproduce the functions u_α^- and u_α^+ on the whole interval $\alpha \in [0,1]$.

For simplicity of notation, we will consider only fuzzy numbers in the form (18) with conditions (19) and (20).

Denote by $\widetilde{\mathbb{F}}_N$ the set of all LU-parametric fuzzy numbers of the form (18) over the same uniform decomposition with N subintervals. We can structure $\widetilde{\mathbb{F}}_N$ by an addition and a scalar multiplication: let $u, v \in \widetilde{\mathbb{F}}_N$ be two LU-parametric fuzzy numbers

$$\begin{aligned} u &= (\alpha_i; u_i^-, \delta u_i^-, u_i^+, \delta u_i^+)_{i=0,1,...,N} \\ v &= (\alpha_i; v_i^-, \delta v_i^-, v_i^+, \delta v_i^+)_{i=0,1,...,N} \end{aligned}$$

then we have

$$\begin{aligned} u + v &= (\alpha_i; u_i^- + v_i^-, \delta u_i^- + \delta v_i^-; u_i^+ + v_i^+, \delta u_i^+ + \delta v_i^+)_{i=0,1,...,N} \\ k \cdot u &= (\alpha_i; k u_i^-, k\delta u_i^-, k u_i^+, k\delta u_i^+)_{i=0,1,...,N} \quad \text{if } k \geq 0 \\ k \cdot u &= (\alpha_i; k u_i^+, k\delta u_i^+, k u_i^-, k\delta u_i^-)_{i=0,1,...,N} \quad \text{if } k < 0. \end{aligned}$$

The gH-difference $w = u \ominus_{gH} v$ or the g-difference $w = u \ominus_g v$ (we use the same w for gH-difference and for g-difference, as they are equal if both exist) in LU-parametric form

$$w = (\alpha_i; w_i^-, \delta w_i^-, w_i^+, \delta w_i^+)_{i=0,1,...,N}$$

can be determined by simple procedures.

The first procedure computes the gH-difference $w = u \ominus_{gH} v$, if it exists, and determines if it is a type (i) (i.e. Hukuhara difference such that $u = v + w$) or a type (ii) gH-difference (i.e. $v = u - w$).

Procedure gHDiff: Compute gH-difference and its type

Given two fuzzy numbers in LU-parametric form
$u = (\alpha_i; u_i^-, \delta u_i^-, u_i^+, \delta u_i^+)_{i=0,1,...,N}$ and $v = (\alpha_i; v_i^-, \delta v_i^-, v_i^+, \delta v_i^+)_{i=0,1,...,N}$
determine if the gH-difference $w = u \ominus_{gH} v$ exists in one of the two forms (i) or (ii) and, if it exists, computes its LU-parametric form
$w = (\alpha_i; w_i^-, \delta w_i^-, w_i^+, \delta w_i^+)_{i=0,1,...,N}$.
The output variable **type** is as follows:

type $= 1$ if type (i) difference exists;
type $= 2$ if type (ii) difference exists;
type $= 0$ if gH-difference does not exist.

for $i = 0, ..., N$
  $m_i = u_i^- - v_i^-$, $p_i = u_i^+ - v_i^+$, $dm_i = \delta u_i^- - \delta v_i^-$, $dp_i = \delta u_i^+ - \delta v_i^+$
end
$type = 0$
if $m_i \leq p_i$, $dm_i \geq 0$, $dp_i \leq 0$ for all $i = 0, 1, ..., N$ **then** $type = 1$
if $m_i \geq p_i$, $dm_i \leq 0$, $dp_i \geq 0$ for all $i = 0, 1, ..., N$ **then** $type = 2$
if $type = 1$ **then**
  $w_i^- = p_i$, $\delta w_i^- = dp_i$, $w_i^+ = m_i$, $\delta w_i^+ = dm_i$, $i = 0, ..., N$
end
if $type = 2$ **then**
  $w_i^- = m_i$, $\delta w_i^- = dm_i$, $w_i^+ = p_i$, $\delta w_i^+ = dp_i$, $i = 0, ..., N$
end

If conditions (19) and (20) are satisfied for the output $(w_i^-, \delta w_i^-, w_i^+, \delta w_i^+)_{i=0,1,...,N}$ of Procedure gHDiff, then $u \ominus_{gH} v$ exists and $w = u \ominus_{gH} v$. Also we observe that if $u, v \in \widetilde{\mathbb{F}}_N$ are two LU-parametric fuzzy numbers

$$u = (\alpha_i; u_i^-, \delta u_i^-, u_i^+, \delta u_i^+)_{i=0,1,...,N}$$
$$v = (\alpha_i; v_i^-, \delta v_i^-, v_i^+, \delta v_i^+)_{i=0,1,...,N}$$

and if the gH-difference $w = u \ominus_{gH} v$ exists, then it is an LU-parametric fuzzy number and it is easy to verify that

$$w_i^- = \min\{u_i^- - v_i^-, u_i^+ - v_i^+\}$$
$$w_i^+ = \max\{u_i^- - v_i^-, u_i^+ - v_i^+\},$$

with associated slopes $\delta w_i^-, \delta w_i^+$, i.e. the procedure described above is correct.

Otherwise, the output $(w_i^-, \delta w_i^-, w_i^+, \delta w_i^+)_{i=0,1,...,N}$ is to be adjusted to obtain the g-difference.

Procedure gDiff: Compute g-difference

Given the output $type = 0$ of procedure **gHDiff**
compute the g-difference $w = u \ominus_{gH} v$ in LU-parametric form
$w = (\alpha_i; w_i^-, \delta w_i^-, w_i^+, \delta w_i^+)_{i=0,1,\dots,N}$.

for $i = 0, \dots, N$
    $m_i = u_i^- - v_i^-$, $p_i = u_i^+ - v_i^+$,
    $dm_i = \delta u_i^- - \delta v_i^-$, $dp_i = \delta u_i^+ - \delta v_i^+$
end
for $i = 0, \dots, N$
    if $m_i = p_i$
        $w_i^- = m_i$, $\delta w_i^- = \max(0, dm_i, dp_i)$
        $w_i^+ = p_i$, $\delta w_i^+ = \min(0, dm_i, dp_i)$
    elseif $m_i < p_i$
        $w_i^- = m_i$, $\delta w_i^- = \max(0, dm_i)$
        $w_i^+ = p_i$, $\delta w_i^+ = \min(0, dp_i)$
    else
        $w_i^- = p_i$, $\delta w_i^- = \max(0, dp_i)$
        $w_i^+ = m_i$, $\delta w_i^+ = \min(0, dm_i)$
    end
end
for $i = N - 1, \dots, 0$
    if $w_i^- \geq w_{i+1}^-$ **then** $w_i^- = w_{i+1}^-, \delta w_i^- = 0, \delta w_{i+1}^- = 0$
    if $w_i^+ \leq w_{i+1}^+$ **then** $w_i^+ = w_{i+1}^+, \delta w_i^+ = 0, \delta w_{i+1}^+ = 0$
end

From Proposition 7 we obtain immediately that for any two LU-parametric fuzzy numbers $u, v \in \widetilde{\mathbb{F}}_N$ the g-difference $w = u \ominus_g v$ is an LU-parametric fuzzy number and

$$
\begin{aligned}
w_i^- &= \inf D_i, \\
w_i^+ &= \sup D_i, \text{ where} \\
D_i &= \{u_j^- - v_j^- | j \geq i\} \cup \{u_j^+ - v_j^+ | j \geq i\},
\end{aligned} \tag{21}
$$

with the corresponding slopes being set to 0 whenever $w_i^- = w_{i+1}^-$ or $w_i^+ = w_{i+1}^+$, i.e., the algorithm described above is correct.

Let us observe also that if the LU-parametric representation of a fuzzy-valued function $f : [a, b] \longrightarrow \mathbb{R}_{\mathcal{F}}$ is obtained by representing each $f(x) \in \mathbb{R}_{\mathcal{F}}$ as in (18); Lemma 22 is useful to characterize the gH-differentiability of a fuzzy-valued function $f :]a, b[\rightarrow \mathbb{R}_{\mathcal{F}}$ defined in terms of its α-cuts $[f(x)]_\alpha = [f_\alpha^-(x), f_\alpha^+(x)]$.

Based on the results established in [27], [28], when both $f_\alpha^-(x)$ and $f_\alpha^+(x)$ are differentiable w.r.t. x for all α's, then the α-cuts of the gH-derivative of f are

$$f'_{gH}(x) = [\min\{(f_\alpha^-)'(x), (f_\alpha^+)'(x)\}, \max\{(f_\alpha^-)'(x), (f_\alpha^+)'(x)\}]$$

provided that the two functions $(f'_{gH}(x))_\alpha^- = \min\{(f_\alpha^-)'(x), (f_\alpha^+)'(x)\}$ and $(f'_{gH}(x))_\alpha^+ = \max\{(f_\alpha^-)'(x), (f_\alpha^+)'(x)$ define (w.r.t. α) a fuzzy number.

As $f_\alpha^-(x)$ and $f_\alpha^+(x)$ define the α-cuts of the fuzzy number $f(x)$ for each x, clearly they are monotonic and almost everywhere differentiable w.r.t. α and satisfy the conditions of Lemma 22. Assume, for simplicity of presentation, that each function $\alpha \longrightarrow f_\alpha^-(x)$ and $\alpha \longrightarrow f_\alpha^+(x)$ is differentiable w.r.t. α.

Notation: We will use the following notations: $\delta f_\alpha^-(x) = \frac{\partial}{\partial\alpha} f_\alpha^-(x)$, $\delta f_\alpha^+(x) = \frac{\partial}{\partial\alpha} f_\alpha^+(x)$, $(f_\alpha^-)'(x) = \frac{\partial}{\partial x} f_\alpha^-(x)$, $(f_\alpha^+)'(x) = \frac{\partial}{\partial x} f_\alpha^-(x)$, and, for short, given a fuzzy valued function $f(x)$, we will denote by $\delta f(x)$ the pairs of functions $(\delta f_\alpha^-(x), \delta f_\alpha^+(x))_{\alpha\in[0,1]}$; at $\alpha = 0$ and $\alpha = 1$, $\delta f(x)$ contains the right and left derivative w.r.t. α.

We will assume that the following equalities hold for the mixed derivatives:

$$(\delta f_\alpha^-)'(x) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial\alpha} f_\alpha^-(x)\right) = \frac{\partial}{\partial\alpha}\left(\frac{\partial}{\partial x} f_\alpha^-(x)\right) = \delta\left((f_\alpha^-)'(x)\right) \tag{22}$$

$$(\delta f_\alpha^+)'(x) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial\alpha} f_\alpha^+(x)\right) = \frac{\partial}{\partial\alpha}\left(\frac{\partial}{\partial x} f_\alpha^+(x)\right) = \delta\left((f_\alpha^+)'(x)\right). \tag{23}$$

The following theorem can be proved.

Theorem 23. *Let $f :]a, b[\to \mathbb{R}_\mathcal{F}$ be defined in terms of its α-cuts $[f(x)]_\alpha = [f_\alpha^-(x), f_\alpha^+(x)]$ satisfying conditions (22)-(23). Then*
1. f is (i)-gH-differentiable at x if and only if we have

$$(i)\begin{cases} (f_1^-)'(x) \le (f_1^+)'(x) \\ (\delta f_\alpha^-)'(x) \ge 0, \forall\alpha \in [0,1] \\ (\delta f_\alpha^+)'(x) \le 0, \forall\alpha \in [0,1] \end{cases}$$

2. $\quad f$ *is (ii)-gH-differentiable at* x *if and only if*

$$(ii)\begin{cases}(f_1^+)'(x) \le (f_1^-)'(x) \\ (\delta f_\alpha^-)'(x) \le 0, \forall\alpha \in [0,1] \\ (\delta f_\alpha^+)'(x) \ge 0, \forall\alpha \in [0,1]\end{cases}$$

PROOF. *The proof is obtained by using Lemma 22 to the families of intervals* $[(f_\alpha^-)'(x), (f_\alpha^+)'(x)]$ *for (i); and* $[(f_\alpha^+)'(x), (f_\alpha^-)'(x)]$ *for (ii).*

Remark 24. A sufficient condition for the equality between the mixed partial derivatives of $f_\alpha^-(x), f_\alpha^+(x)$ regarded as bivariate functions of x and α is that these functions are twice continuously differentiable on their domain.

As we have remarked for the rational shape function (12), (monotonic) linear and quadratic shape functions $f_\alpha^-(x)$, $f_\alpha^+(x)$ are naturally represented, with respect to $\alpha \in [0,1]$, by the trivial decomposition with only two points $0 = \alpha_0 < \alpha_1 = 1$ ($N = 1$) so that the LU-parametrization of linear (or quadratic) $f(x)$ is

$$f(x) = (f_0^-(x), \delta f_0^-(x), f_0^+(x), \delta f_0^+(x);\ f_1^-(x), \delta f_1^-(x), f_1^+(x), \delta f_1^+(x))$$

with

$$\begin{array}{c} f_0^-(x) \le f_1^-(x) \le f_1^+(x) \le f_0^+(x) \text{ and} \\ \delta f_i^-(x) \ge 0, \delta f_i^+(x) \le 0,\ i = 0,1. \end{array}$$

Note that the slopes $\delta f_i^-(x)$ and $\delta f_i^+(x)$ are the derivatives of $f_\alpha^-(x)$ and $f_\alpha^+(x)$ with respect to α at $\alpha = 0$ and $\alpha = 1$.

The (i)-gH-derivative is (we omit here the reference to x)

$$\begin{array}{l} f'_{gH} = ((f_0^-)', (\delta f_0^-)', (f_0^+)', (\delta f_0^+)'; (f_1^-)', (\delta f_1^-)', (f_1^+)', (\delta f_1^+)') \\ \quad \text{if} \\ (f_0^-)' \le (f_1^-)' \le (f_1^+)' \le (f_0^+)' \\ (\delta f_0^-)' \ge 0,\ (\delta f_1^-)' \ge 0 \ \text{ and } \ (\delta f_0^+)' \le 0,\ (\delta f_1^+)' \le 0. \end{array}$$

and the (ii)-gH-derivative is

$$\begin{array}{l} f'_{gH} = ((f_0^+)', (\delta f_0^+)', (f_0^-)', (\delta f_0^-)'; (f_1^+)', (\delta f_1^+)', (f_1^-)', (\delta f_1^-)') \\ \quad \text{if} \\ (f_0^-)' \ge (f_1^-)' \ge (f_1^+)' \ge (f_0^+)' \\ (\delta f_0^-)' \le 0,\ (\delta f_1^-)' \le 0 \ \text{ and } \ (\delta f_0^+)' \ge 0,\ (\delta f_1^+)' \ge 0. \end{array}$$

In particular, any triangular (or trapezoidal) fuzzy-valued function is such that $\delta f_0^-(x) = \delta f_1^-(x) = f_1^-(x) - f_0^-(x)$ and $\delta f_0^+(x) = \delta f_1^+(x) = f_1^+(x) - f_0^+(x)$ and the four values $f_0^-(x) \leq f_1^-(x) \leq f_1^+(x) \leq f_0^+(x)$ are sufficient to fully define it; in fact, from the identity $p_{rat}(\alpha; 1, 1) = \alpha$, we find that equations (15-16) become

$$f_\alpha^\pm(x) = f_0^\pm(x) + \alpha(f_1^\pm(x) - f_0^\pm(x)),\ \alpha \in [0, 1].$$

As a consequence, the gH-derivative of a triangular or a trapezoidal fuzzy-valued function is itself triangular or trapezoidal and its LU-parametric representation with the trivial decomposition is exact.

Example 25. Consider the fuzzy valued function $f : [-2, 2] \rightarrow \mathbb{R}_\mathcal{F}$ having triangular values as outputs:

$$f(x) = \left(\frac{x^3}{3}, \frac{x^3}{3} + x + 3, \frac{2x^3}{3} + 4\right);$$

its level sets are

$$\begin{aligned} f_\alpha^-(x) &= \frac{x^3}{3} + \alpha(x + 3) \\ f_\alpha^+(x) &= (2 - \alpha)\frac{x^3}{3} + x\alpha + 4 - \alpha; \end{aligned}$$

and

$$\begin{aligned} \left(f_\alpha^-\right)'(x) &= x^2 + \alpha \\ \left(f_\alpha^+\right)'(x) &= (2 - \alpha)x^2 + \alpha; \end{aligned}$$

the derivatives w.r.t. α are constant in α

$$\begin{aligned} \delta f_\alpha^-(x) &= x + 3 \\ \delta f_\alpha^+(x) &= -\frac{x^3}{3} + x - 1 \end{aligned}$$

and

$$\begin{aligned} (\delta f_\alpha^-)'(x) &= 1 \\ (\delta f_\alpha^+)'(x) &= -x^2 + 1. \end{aligned}$$

We observe that on the intervals $[-2, -1]$ and $[1, 2]$ the function is gH-differentiable, namely it is Hukuhara differentiable. In the interval $[-1, 1]$ it is not gH-differentiable but it is g-differentiable (see Figs. 1, 2). The level sets in the figures are between pairs of curves one blue (lower) and green (upper) with innermost being the curves that delimit the 1-level set and outermost pair will be delimiters for the support.

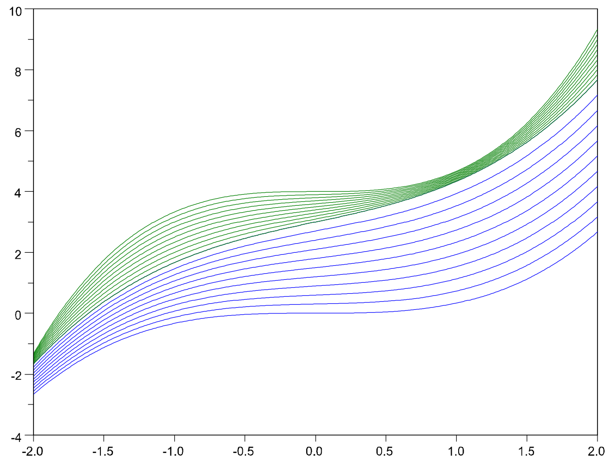

Figure 1. Level sets of a fuzzy valued function in Example 25

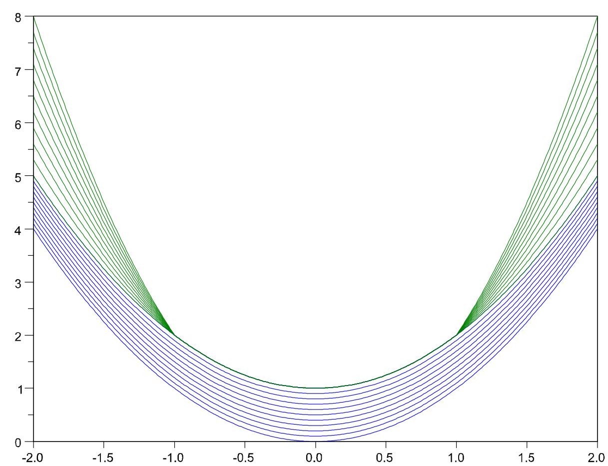

Figure 2. The g-derivative of the function in Example 25

Example 26. Let us consider one more simple example that illustrates the g-differentiability concept. Let $f : [-1, 1] \to \mathbb{R}_{\mathcal{F}}$ having triangular values

$$f(x) = \left(0, x^2, x^2 + 1\right).$$

Then the functions giving the endpoints of the level sets are

$$\begin{aligned} f_\alpha^-(x) &= \alpha x^2 \\ f_\alpha^+(x) &= (\alpha + 1)x^2 + 1 - \alpha; \end{aligned}$$

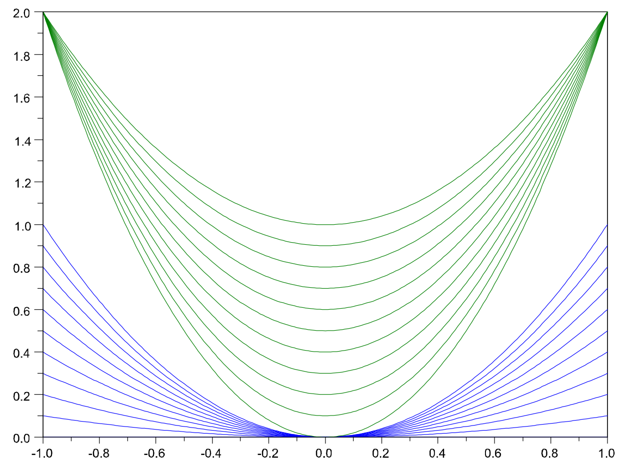

Figure 3. Level sets of a fuzzy valued function in Example 26

having

$$\left(f_\alpha^-\right)'(x) = 2\alpha x$$
$$\left(f_\alpha^+\right)'(x) = 2(\alpha+1)x;$$

the derivatives w.r.t. α are

$$\delta f_\alpha^-(x) = x^2$$
$$\delta f_\alpha^+(x) = x^2 - 1$$

and

$$(\delta f_\alpha^-)'(x) = 2x$$
$$(\delta f_\alpha^+)'(x) = 2x.$$

We observe that the function is not gH-differentiable but it is g-differentiable (see Figs. 3, 4).

In the next propositions we analyze the gH-differentiability and the g-differentiability under the LU-parametric representation.

Proposition 27. *Let $f:[a,b]\longrightarrow \mathbb{R}_\mathcal{F}$ be represented in LU-parametric form*

$$f(x) = \left(\alpha_i; f_i^-(x), \delta f_i^-(x), f_i^+(x), \delta f_i^+(x)\right)_{i=0,\dots,N};$$

assume that for $i = 0,1,...,N$ the functions $f_i^-(x), \delta f_i^-(x),\ f_i^+(x), \delta f_i^+(x)$ are differentiable at $x = x_0$. Then

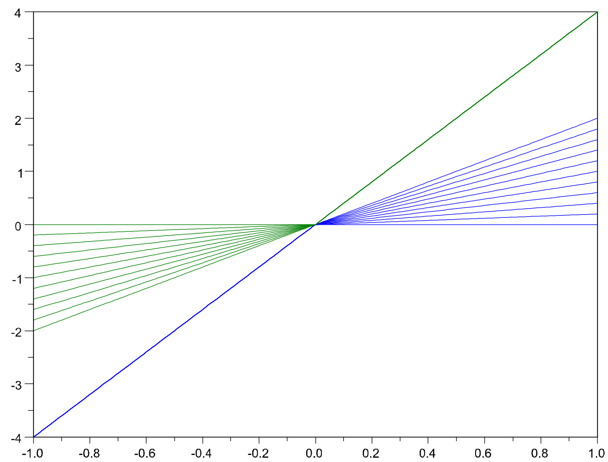

Figure 4. The g-derivative of the function in Example 26

(1) f is (i)-gH-differentiable at x_0 if and only if the following is a fuzzy number

$$w = \left(\alpha_i; \left(f_i^-\right)'(x_0), \left(\delta f_i^-\right)'(x_0), \left(f_i^+\right)'(x_0), \left(\delta f_i^+\right)'(x_0)\right)_{i=0,\dots,N}; \tag{24}$$

(2) f is (ii)-gH-differentiable at x_0 if and only if the following is a fuzzy number

$$w = \left(\alpha_i; \left(f_i^+\right)'(x_0), \left(\delta f_i^+\right)'(x_0), \left(f_i^-\right)'(x_0), \left(\delta f_i^-\right)'(x_0)\right)_{i=0,\dots,N}. \tag{25}$$

In any case, we have $f'_{gH}(x_0) = w$.

PROOF. *Direct calculation by Theorem 13.*

The following proposition is also immediate.

Proposition 28. *Let $f : [a,b] \longrightarrow \mathbb{R}_{\mathcal{F}}$ be represented in LU-parametric form*

$$f(x) = \left(\alpha_i; f_i^-(x), \delta f_i^-(x), f_i^+(x), \delta f_i^+(x)\right)_{i=0,\dots,N};$$

assume that for $i = 0, 1, \dots, N$ the functions $f_i^-(x)$, $\delta f_i^-(x)$, $f_i^+(x)$, $\delta f_i^+(x)$ are differentiable at $x = x_0$. Then f is g-differentiable at x_0 and the fuzzy number $f'_g(x_0)$ has the following LU-parametrization

$$f'_g(x_0) = \left(\alpha_i; w_i^-, \delta w_i^-, w_i^+, \delta w_i^+\right)_{i=0,\dots,N},$$

*where $w_i^-, \delta w_i^-, w_i^+, \delta w_i^+$ are obtained by procedures **gHDiff** and **gDiff** applied with input data (the derivatives $()'$ are with respect to x):*

$$m_i = (f_i^-)'(x_0), p_i = (f_i^+)'(x_0),$$

$$dm_i = (\delta f_i^-)'(x_0), dp_i = (\delta f_i^+)'(x_0).$$

Remark 29. The last two propositions can be used to determine the type of a switching point x_0, by running procedure **gHDiff** at two points $x_0 - \delta$ and $x_0 + \delta$ with a sufficiently small $\delta > 0$. We can have several cases, according to the output value of parameter **type** from routine **gHDiff**; denoting $type_L$ and $type_R$ the **type** of g-derivative at points $x_0 - \delta$ and $x_0 + \delta$, respectively (assuming that $type_L \neq type_R$), we have the following combinations:

- if $type_L = 1$ and $type_R = 2$, then x_0 is a *type-I switch*;
- if $type_L = 2$ and $type_R = 1$, then x_0 is a *type-II switch*;
- if $type_L = 1$ and $type_R = 0$, then x_0 is a *switch* from (i)-gH-differentiability to g-differentiability;
- if $type_L = 2$ and $type_R = 0$, then x_0 is a *switch* from (ii)-gH-differentiability to g-differentiability;
- if $type_L = 0$ and $type_R = 1$, then x_0 is a *switch* from g-differentiability to (i)-gH-differentiability;
- if $type_L = 0$ and $type_R = 2$, then x_0 is a *switch* from g-differentiability to (ii)-gH-differentiability.

It is also simple to determine the type of transitional region for an interval $[x_1, x_2]$ where f is g-differentiable of type 0: we compute $type_L$ at point $x_1 - \delta$ and $type_R$ at point $x_2 + \delta$ and we compare $type_L$ with $type_R$ in the appropriate way.

4 LU-parametric approximation of fuzzy g-derivative

For general fuzzy-valued functions, the LU-parametric form (15-16) can be used as an approximation tool. As discussed e.g. in [8] and [29], the quality of the approximation of general fuzzy numbers is increased by refining the decomposition of interval $[0, 1]$ from the trivial $\{\alpha_0 = 0, \alpha_1 = 1\}$ with two points, to $\{0 = \alpha_0 < \alpha_1 < ... < \alpha_N = 1\}$ with $N + 1$ points. In this section we will illustrate some computational results, to show that the approximation error reduces very rapidly by increasing the number N of subintervals in the decomposition.

The fuzzy valued function in the following example has points where it is (i)-gH differentiable, points where it is (ii)-gH differentiable, and points where it is g-differentiable (see [4]).

Consider the fuzzy valued function $[f(x)]_\alpha = [f_\alpha^-(x), f_\alpha^+(x)]$, $x \in [0, 2\pi]$ with

$$f_\alpha^-(x) = \frac{x^2}{40} + \frac{(3\alpha^2 - 2\alpha^3)\sin^2(x)}{20}$$
$$f_\alpha^+(x) = \frac{x^2}{40} + \frac{(2 - 3\alpha^2 + 2\alpha^3)\sin^2(x)}{20}$$

At $x \in \{0, \pi, 2\pi\}$, function $f(x)$ has a crisp value. Observe that the x-derivatives $(f_\alpha^-)'(x)$ and $(f_\alpha^+)'(x)$ are cubic functions of $\alpha \in [0,1]$ and the slopes $\delta(f_\alpha^-)'(x)$ and $\delta(f_\alpha^+)'(x)$ are computed easily. The LU-parametric form of $f_g'(x)$ on a uniform decomposition (N subintervals) $\alpha_i = i/N$, $i = 0, 1, ..., N$ is obtained by the application of Proposition 28; so, it can be computed as soon as the four functions $(f_\alpha^-)'(x)$, $(f_\alpha^+)'(x)$, $\delta(f_\alpha^-)'(x)$ and $\delta(f_\alpha^+)'(x)$ are available (for each x) at the $N+1$ points $\alpha_i = i/N$, $i = 0, 1, ..., N$. With the LU-form available, we can finally approximate the α-cuts of the g-derivative $f_g'(x)$ by equations (15-16). To see how the approximation improves by increasing N, consider the results of Table 1. The average absolute and relative errors are determined by comparing the exact α-cuts $[f_g'(x)]_\alpha$ and the α-cuts $[f_{LU}'(x)]_\alpha$ of the approximated g-derivative.

The exact and approximated g-derivatives are computed at $P = 201$ uniform points $x_k \in [0, 2\pi]$ and $M = 101$ uniform values $\alpha_j \in [0,1]$; denote by $[z_{k,j}^-, z_{k,j}^+]$ and by $[\widetilde{z}_{k,j}^-, \widetilde{z}_{k,j}^+]$ the α-cuts $[f_g'(x_k)]_{\alpha_j}$ and $[f_{LU}'(x_k)]_{\alpha_j}$, respectively. The error measures reported in Table 1, for different values of N are the average absolute error $AERdF$, the percentage relative mean squared error $\%RMSE$ and the percentage relative mean absolute error $\%RMAE$; they are defined, for T exact values X_t and approximated $\widetilde{X_t}$, $t = 1, ..., T$, by the following expressions

$$AERdF = \frac{1}{T}\sum_{t=1}^{T}\left(|X_t - \widetilde{X}_t|\right)$$

$$\%RMSE = 100\sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(\frac{X_t - \widetilde{X}_t}{X_t}\right)^2},$$

and

$$\%RMAE = 100\frac{1}{T}\sum_{t=1}^{T}\left|\frac{X_t - \widetilde{X}_t}{X_t}\right|.$$

Table 1.	Approximation errors for LU-parametric g-difference and different N					
N	1	2	4	8	10	20
$AERdF$	0.1×10^{-2}	0.22×10^{-3}	0.17×10^{-4}	0.51×10^{-5}	0.60×10^{-6}	0.47×10^{-9}
$\%RMSE$	1.89%	0.42%	0.046%	0.004%	0.002%	0.0002%
$\%RMAE$	1.11%	0.23%	0.019%	0.002%	0.0007%	0.00005%

We see that three α-cuts ($N = 2$) are sufficient for an error less than 1%, and 8 intervals give a relative average error of the order $\frac{2}{100000}$.

5 Conclusions and further work

The g-differentiability introduced by the authors in [4], is a very general derivative concept, being also practically applicable. In this paper, following the same the research direction, we investigate the LU-parametric representation of fuzzy numbers in the setting of g-differentiability and show necessary and sufficient conditions for types of generalized fuzzy differentiability (e.g. (i)-gH-differentiability and (ii)-gH-differentiability). We also present some computational procedures to determine the LU-parametric form of the fuzzy g-derivative and to establish its type. We conclude that the LU-parametrization is a promising way to improve computations in terms of speed and approximation quality.

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