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- Fabio Tramontana (U. Ancona)
- Laura Gardini (U. Urbino)
- Toni Puu (CERUM, Umeå University)

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Mathematical Properties of a Combined Cournot-Stackelberg model

Fabio Tramontana[1], **Laura Gardini**
Department of Economics and Quantitative Methods,
University of Urbino (Italy).
e-mail: laura.gardini@uniurb.it, f.tramontana@univpm.it

Tönu Puu
CERUM, Umeå University, SE-90187 Umeå, Sweden

Abstract

The object of this work is to perform the global analysis of a new duopoly model which couples the two points of view of Cournot and Stackelberg. The Cournot model is assumed with isoelastic demand function and unit costs. The coupling leads to discontinuous reaction functions, whose bifurcations, mainly border collision bifurcations, are investigates as well as the global structure of the basins of attraction. In particular, new properties are shown, associated with the introduction of horizontal branches, which differ significantly when the constant value is zero or positive and small. The good behavior of the model with positive constant is proved, leading to stable cycles of any period.

Keywords: Cournot-Stackelberg duopoly; Isoelastic demand function; Discontinuous reaction functions; Multistability; Border-collision bifurcations

JEL-Classification: C15, D24, D43

[1] Corresponding author. Department of Economics and Quantitative Methods, Via Saffi 42, University of Urbino (Italy). e-mail: f.tramontana@univpm.it

1 Introduction.

About a Century after Cournot first invented duopoly theory (Cournot 1838), Heinrich von Stackelberg (1934) proposed one of the most original developments of it. Cournot defined an iterative dynamical system in terms of reaction functions, which stated the best decisions for maximizing profits of each competitor given the decision of the other. As only expectations of the latter, based on past observations, could be formed; in the simplest case just extrapolating from the preceding period, the reaction functions were easily interpreted as iterative systems. If the reaction functions were taken as a simultaneous system of equations, they in stead determined equilibrium states, Cournot equilibria. Cournot's theory hence came in both a dynamic and a static variant. A Cournot equilibrium defined by the static system could be either stable or unstable when studied in terms of the dynamic process.

Stackelberg proposed that either competitor could learn the reaction function of the other, hence becoming a "leader" treating the other as "follower", and so solve, not a partial optimization problem, conditional upon the decision by the competitor, but an optimization problem that took explicit account of such partial optimization by the competitor. If the other competitor actually behaved as assumed, i.e., followed the Cournot reaction function, then a consistent leader/follower pair would be formed. In this way two Stackelberg equilibria were added to the Cournot equilibrium. It is clear that Stackelberg's theory belongs to the Cournot setting, though, unlike this, it only comes in an equilibrium format. It therefore is not clear how one could fit the theories together in one system, Stackelberg's theory is more general as it extends the possibilities for equilibria, but Cournot's is more general as it contains a dynamic perspective. In hindsight it seems a bit surprising that Stackelberg never took a dynamic perspective; if the follower always used the proper reaction function, the leader might have devised a periodic or other changing policy over time in the pursuit of maximum leadership profit. One of the present authors in a recent publication (Puu, 2010) showed how an oscillating sales policy could result in higher leadership profits than sticking to an equilibrium Stackelberg solution constant over time.

However, it is obvious that if one wants to unify the two theories in a dynamic format, then one has to start out from Cournot's theory. One can include the possibility for the duopolists to choose Stackelberg leadership action along with traditional Cournot action in the Cournot dynamic process through some branch condition dependent on the expected profits from Stackelberg leadership as compared to current action using the proper Cournot reaction function each and every time period.

How this could be done was shown by one of the present authors in Puu (2009). This proposed model will be the objective of the present study, as its mathematical properties were not fully analyzed.

It is hence possible to generalize the Cournot dynamic model to include Stackelberg action, and so contain Stackelberg equilibria as additional fixed points, but doing this one encounters a kind of paradox. It is well known that the profits from Stackelberg leadership equilibrium, provided the leader succeeds in keeping the competitor as follower, are higher than the profits for that firm in Cournot equilibrium. But this only holds for the two kinds of *equilibria*. Suppose the competitors are mid in a dynamic process, and the leader has actually succeeded in making the competitor behave as a follower, which is then the best choice for the successful leader in the next period, keeping to Stackelberg leadership, or switching to Cournot action? Paradoxically it is to switch to Cournot action. The simple reason is that the Cournot reaction function was *defined* as the best reply in each market situation.

Hence, if one introduced a comparison between expected current profits from Cournot action to Stackelberg equilibrium profits, the competitors would *never* choose Stackelberg leadership. Is it then better to skip Stackelberg action and always keep to Cournot action? This is not true either, because we know that, at least if the Cournot equilibrium is stable, then the process would converge to this state which yields lower profit than Stackelberg leadership. As a matter of fact, with the model proposed in Puu (2009), one can see that in any period when the leader is successful, is it better to switch to Cournot action the next period. But, in the period after that, when again Cournot action is defined as the best current choice, would it have been better to stay in Stackelberg equilibrium.

As the competitors would want the maximum of an oscillating profit over time, an endogenous branch condition comparing Stackelberg and Cournot profits would have to include a weighting factor, stating, for instance, that the competitors switch to Stackelberg action whenever Stackelberg (equilibrium) profits exceed, say 75 percent of current expected Cournot profits, not 100 percent, because they never do. This is the meaning of the weighting factor.

After describing the model formulation, it will be studied purely mathematically, as the economics discussion in Puu (2009) was rather complete. It is worth noting that the resulting reaction functions are piecewise smooth and discontinuous, and we shall combine mathematical results from the continuous generic duopoly model in Bischi et al. (2000) and the discontinuous model in Tramontana et al. (2008). The main result with respect to Cournot models already known in the literature (see Kopel (2006), Puu (1998), Ahmed et al. (2000), Agiza and Elsadany (2004), Agliari (2006), Agliari et al. (2006a, 2006b), Angelini et al. (2009), Tramontana (2010) to cite a few), which can also have complex dynamics, is that the Cournot-Stackelberg duopoly model here considered has always coexisting stable cycles of low periods.

The techniques here used, can be extended also to several other models, proposed for example in Puu and Sushko (2002) and Bischi et al. (2009) as well as in many other duopoly or oligopoly piecewise smooth models. Moreover, following Tramontata et al. (2010), a new element is proposed also in the model formulation, to avoid extinction in the productions.

So the plan of the work is as follows. In Section 2 we shall recall the model, considering the case in which the reaction functions are defined with a zero branch, in order to avoid negative productions which have no economic meaning (as usual, since Puu (1991)). The global dynamics associated with it are described in Section 3: the new elements are due to the horizontal branches and the discontinuity points of the reaction functions, leading to border collision bifurcations. We shall see that the dynamics are stable, converging either to equilibria or to cycles of low period. However, in extreme situations, also when the Cournot equilibrium is locally stable many trajectories may be mapped into the invariant coordinate axes (with periodic zero productions). Thus in Section 4 we shall consider the modified model in which the zero branch of the reaction functions is changed into a small positive constant value. This economically plausible change leads to dynamics which are always positive. The states previously convergent to the axes now are convergent to some cycle in the positive phase space. As we shall see, also when the Cournot fixed point is locally unstable the dynamics are mainly convergent to a unique superstable cycle, whose period may be any integer number, depending on the parameters and on the small constant value assumed in the model. Section 5 concludes.

2 Model Setup

2.1 Cournot Reaction Functions

Assume the inverse demand function

$$p = \frac{1}{x+y}, \tag{1}$$

where p denotes market price and x, y denote the outputs of the duopolists. The competitors have constant marginal costs, denoted a, b respectively,so the profits become,

$$U = \frac{x}{x+y} - ax, \tag{2}$$

$$V = \frac{y}{x+y} - by. \tag{3}$$

Putting the derivatives $\frac{\partial U}{\partial x} = 0$ and $\frac{\partial V}{\partial y} = 0$, and solving for x, y, one obtains

$$x' = \sqrt{\frac{y}{a}} - y, \tag{4}$$

$$y' = \sqrt{\frac{x}{b}} - x, \tag{5}$$

which are the reaction functions. The dash represents the next iterate of one competitor given the last observed supply of the other.

As the unimodal reaction functions eventually come down to the axes, and as negative supplies make no sense, a first choice is to replace negative values with a zero branch. The function in (4) returns a negative reply x' if $y > \frac{1}{a}$, and (5) a negative reply y' if $x > \frac{1}{b}$. To avoid this, we put $x' = 0$ whenever $y > \frac{1}{a}$, and $y' = 0$ whenever $x > \frac{1}{b}$. This means reformulating (4)-(5) as follows

$$x' = \begin{cases} \sqrt{\frac{y}{a}} - y, & y \le \frac{1}{a} \\ 0, & y > \frac{1}{a} \end{cases}, \tag{6}$$

$$y' = \begin{cases} \sqrt{\frac{x}{b}} - x, & x \le \frac{1}{b} \\ 0, & x > \frac{1}{b} \end{cases}. \tag{7}$$

As negative supplies would also be related to negative profits it is natural, as remarked above, to assume that after the reaction function comes down to the axis the firm produces nothing. However, once one axis is hit, the system can end up at the origin where the reaction functions also intersect, i.e., at the collusion state. This is, however, forbidden by law in most countries. Further, the reaction functions intersect with infinite slope in the origin, so it is totally unstable, and the system would be thrown away by any slight disturbance. Yet, as we shall see in the next section, solutions involving the zero branches are there and even become stable in a weak Milnor sense. This has never been properly investigated, and it will be done in the next Section.

One can also avoid the origin through stipulating that the duopolists do not actually close down when they cannot make any profit, but keep to some small "epsilon" stand-by output. This assumption was originally introduced in Puu (1991) to the end of keeping the computer from sticking to a totally unstable origin in numerical work, but it makes sense also in terms of substance. The importance of the numerical value of this "epsilon" stand-by output has been investigated for the first time in Tramontana et al. (2010) and in Section 4 we shall consider also its effect in the present Cournot-Stackelberg model.

2.2 Cournot Equilibrium

Putting $x' = x, y' = y$, one can solve for the coordinates of the Cournot equilibrium point

$$x = \frac{b}{(a+b)^2}, \tag{8}$$

$$y = \frac{a}{(a+b)^2}. \tag{9}$$

Substituting back from (8)-(9) in (2)-(3), one further gets the profits of the competitors in the Cournot equilibrium point

$$U = \frac{b^2}{(a+b)^2}, \tag{10}$$

$$V = \frac{a^2}{(a+b)^2}. \tag{11}$$

Note that (10)-(11) are the profits in Cournot equilibrium. During the dynamic Cournot process (4)-(5), profits can be considerably higher. To calculate these temporary profits, just substitute from (4) into (2), and from (5) into (3), to obtain

$$U = (1 - \sqrt{ay})^2, \tag{12}$$

$$V = \left(1 - \sqrt{bx}\right)^2. \tag{13}$$

2.3 Stackelberg Equilibria

As mentioned, Stackelberg action only comes in an equilibrium format. This also implies that only Stackelberg equilibrium profits make sense, and it is these that will be compared to the Cournot temporary profits (12)-(13).

According to Stackelberg's idea, the competitor controlling x can take the reaction function (5) of the other for given, and substitute it in its own proper profit function (2), to obtain

$$U = \sqrt{bx} - ax.$$

Putting $\frac{dU}{dx} = 0$, and solving, one gets

$$x = \frac{b}{4a^2}. \tag{14}$$

The corresponding value of y, provided that firm really adheres to its Cournot reaction function, is obtained through substituting (14) in (5) and equals

$$y = \frac{2a - b}{4a^2}. \tag{15}$$

Using (14)-(15) in (3), the Stackelberg leadership profit can be easily calculated

$$U = \frac{b}{4a}. \tag{16}$$

Similarly, the second firm can try leadership, substituting (4) in (3), and maximizing

$$V = \sqrt{ay} - by,$$

to obtain

$$y = \frac{a}{4b^2}. \tag{17}$$

The corresponding Cournot response of the first firm would then be

$$x = \frac{2b - a}{4b^2}, \tag{18}$$

and the Stackelberg leadership profit

$$V = \frac{a}{4b}. \tag{19}$$

It is easy to check that Stackelberg leadership profits always exceed the respective Cournot equilibrium profits, i.e., (16) is higher than (10), and (19) higher than (11).

2.4 Reswitching to Cournot Action

Suppose now that the firm controlling x successfully established Stackelberg leadership, i.e., chose $x = \frac{b}{4a^2}$, whereas the firm controlling y responded with $y = \frac{2a-b}{4a^2}$. Is it better for the first firm to keep to leadership and obtain the profit $U = \frac{b}{4a}$, or switch to the reaction function, obtaining the profit $U = \left(1 - \sqrt{ay}\right)^2$, where $y = \frac{2a-b}{4a^2}$. Substituting, $U = \left(1 - \frac{1}{2}\sqrt{2 - \frac{b}{a}}\right)^2$, which is always higher than $U = \frac{b}{4a}$, unless $a = b$, in which case the expressions are equal. The conclusion is that it always pays even for thee successful leader to switch to the reaction function for one period. Continuing the calculations, which become too messy to be reproduced, show that this situation only lasts for one period. The best choice for each single period is to keep to Cournot action, but already the next period profits become lower than in Stackelberg leadership, and if the Cournot equilibrium is stable, the successive process will lead to this equilibrium which yields the long run equilibrium profit $U = \frac{b^2}{(a+b)^2}$, which definitely is lower than $U = \frac{b}{4a}$.

So, it seems that at some stage the leader firm will try to reswitch to Stackelberg action, comparing expected Cournot profits $\left(1 - \sqrt{ay}\right)^2$ to leadership profits $\frac{b}{4a}$, but the condition $\left(1 - \sqrt{ay}\right)^2 > \frac{b}{4a}$ is not suitable, because it never holds. It is necessary to include a weighting factor in this comparison. There is nothing absurd in this, because the competitors will have to maximize profits over some longer period, and may establish such a weighting factor from actual experience.

2.5 The Jump Condition

Denoting this weighting factor by k, and recalling that both competitors can attempt Stackelberg leadership actions, one has the following two conditions for keeping to Cournot reaction functions.

$$(1 - \sqrt{ay})^2 \geq k\frac{b}{4a}, \tag{20}$$

$$\left(1 - \sqrt{bx}\right)^2 \geq k\frac{a}{4b}. \tag{21}$$

The value of the parameter k indicates how adventurous the competitors are at attempting a jump to leadership action. It was argued that it might exceed unity if the competitors were ever to try leadership, but some competitors might never do, so any positive values for k, here for simplicity taken equal for the competitors, make sense.

2.6 The Map

Recalling the zero branches, it is now possible to specify the map,

$$x' = \begin{cases} \sqrt{\frac{y}{a}} - y, & y \leq \frac{1}{a} \ \& \ \left(1 - \sqrt{ay}\right)^2 \geq k\frac{b}{4a} \\ \frac{b}{4a^2}, & y \leq \frac{1}{a} \ \& \ \left(1 - \sqrt{ay}\right)^2 < k\frac{b}{4a} \\ 0 & y > \frac{1}{a} \end{cases}, \tag{22}$$

$$y' = \begin{cases} \sqrt{\frac{x}{b}} - x, & x \leq \frac{1}{b} \ \& \ \left(1 - \sqrt{bx}\right)^2 \geq k\frac{a}{4b} \\ \frac{a}{4b^2}, & x \leq \frac{1}{b} \ \& \ \left(1 - \sqrt{bx}\right)^2 < k\frac{a}{4b} \\ 0, & x > \frac{1}{b} \end{cases}. \tag{23}$$

This can accommodate the Cournot equilibrium as well as both Stackelberg equilibria.

3 The basic Cournot-Stackelberg model.

For convenience the above map can be reformulated as map T, $T(x,y) = (x',y')$ defined as follows:

$$x' = f(y) = \begin{cases} f_C(y) = \sqrt{\frac{y}{a}} - y & if \;\; y \le d_y = \frac{1}{a}\left(1 - \frac{1}{2}\sqrt{kr}\right)^2 \\ f_S(y) = \frac{b}{4a^2} & if \;\; d_y < y \le \frac{1}{a} \\ 0 & if \qquad y > \frac{1}{a} \end{cases} \tag{24}$$

$$y' = g(x) = \begin{cases} g_C(x) = \sqrt{\frac{x}{b}} - x & if \;\; x \le d_x = \frac{1}{b}\left(1 - \frac{1}{2}\sqrt{\frac{k}{r}}\right)^2 \\ g_S(x) = \frac{a}{4b^2} & if \;\; d_x < x \le \frac{1}{b} \\ 0 & if \qquad x > \frac{1}{b} \end{cases} \tag{25}$$

where we have used the ratio $r = \frac{b}{a}$. It is immediate to see that the parameters (a,b), which are the usual coefficients in the Cournot duopoly game, also determine the constant value for the Stackelberg regime. While the parameter k only influences the constraints, the interval of action of the Cournot branch or of the Stackelberg value. It follows that the equilibria and the cycles of the model (both the coordinates of the periodic points and their local stability) only depend on the two parameters (a,b). However, the existence of cycles is also determined by the position of the discontinuity points of the functions, and thus depend on $d_x(r,k)$ and $d_y(r,k)$. The related conditions of appearance/disappearance of cycles are called border collision bifurcations (BCB henceforth), because strictly related with the positions of the discontinuity points. It is also worth to mention that there is a symmetry in the model, given by $T(x,y,a,b) = T(y,x,b,a)$ leading to a symmetric structure of the bifurcation curves in the two dimensional parameter plane (a,b), with respect to the line $a = b$. It follows that it is suitable to reduce of one unit the number of the parameters keeping, besides k, the unique parameter $r = \frac{b}{a}$. This requires a rescaling in the variables: setting $X = bx$ and $Y = ay$ we obtain a two dimensional map which only depends on $(X,Y;r,k)$, $\widetilde{T}(X,Y) = (X',Y')$ (which is clearly topologically conjugated with T) given by:

$$X' = f(Y) = \begin{cases} f_C(Y) = r(\sqrt{Y} - Y) & if \;\; Y \le d_Y = \left(1 - \frac{1}{2}\sqrt{kr}\right)^2 \\ f_S(Y) = \frac{r^2}{4} & if \;\; d_Y < Y \le 1 \\ 0 & if \qquad Y > 1 \end{cases} \tag{26}$$

$$X' = g(X) = \begin{cases} g_C(X) = \frac{1}{r}(\sqrt{X} - X) & if \quad X \leq d_X = \left(1 - \frac{1}{2}\sqrt{\frac{k}{r}}\right)^2 \\ g_S(X) = \frac{1}{4r^2} & if \quad d_X < X \leq 1 \\ 0 & if \qquad X > 1 \end{cases} \tag{27}$$

As we shall see, and as already known for duopoly games having a second iterate with separate variables (see Bischi et al. (2000)), the peculiarity of the map is the coexistence of stable equilibria. Moreover, the structure of the model leads to stable dynamics: fixed points or cycles of low period. The only dangerous situation occurs when the dynamics disappear, i.e. the trajectories are leading to extinction of the game, due to the zero branches in the reaction functions. This second undesired behavior will be corrected in the next section with the modified model.

Let us start our analysis noting a first condition that must be satisfied. In order to have a meaningful model the parameters must give positive discontinuity points, $d_Y > 0$ and $d_X > 0$, which leads to

$$k < \min\left\{4r \ , \ \frac{4}{r}\right\} \quad , \quad r = \frac{b}{a} \tag{28}$$

or, equivalently:

$$\frac{k}{4} < r < \frac{4}{k} \tag{29}$$

and the conditions $d_Y < 1$ and $d_X < 1$ are always satisfied. As proved in Bischi et al. (2000), the dynamic behavior of a duopoly model is governed by the one-dimensional map $X' = F(X) = f(g(X))$ that in our case is discontinuous.

An example is shown in Fig.1a ($r = 3$, $k = 0.6$), where we can see that the function $F(x)$ has two discontinuity points, in $X = d_X$ and in $X = 1$. The first two branches of the function $F(X)$ are given by

$$F_{CC}(X) = f_C(g_C(X)) \ , \tag{30}$$

$$F_{CS}(X) = f_C(g_S(X)) = f_C(\frac{1}{4r^2}) = \frac{1}{2}(1 - \frac{1}{2r}) \tag{31}$$

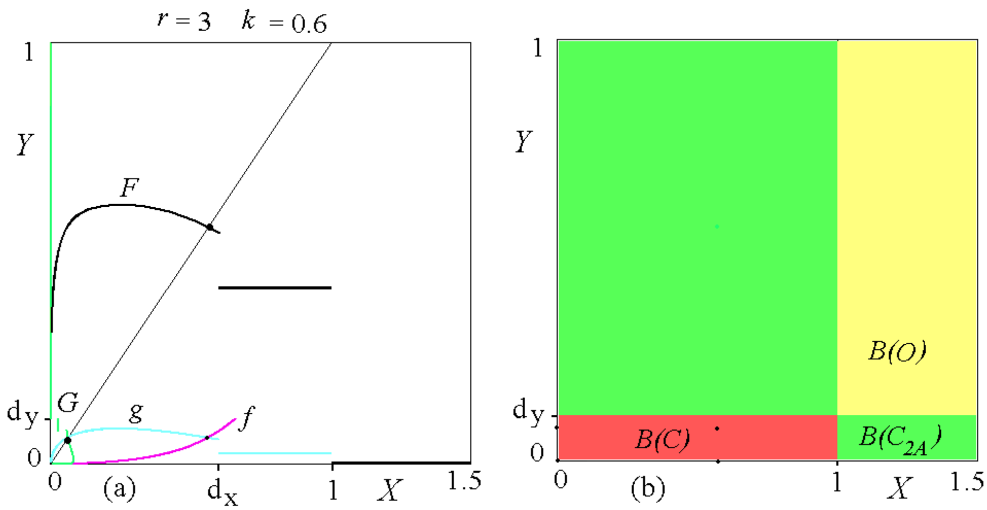

Fig.1 $r = 3$, $k = 0.6$. Reaction functions in (a). Basins of attraction in (b).

Considering the case shown in Fig.1a, from the existence of two fixed points of $F(X)$, which are the origin $O = (0, 0)$ (locally unstable) and the Cournot point $C = (X_C^*, Y_C^*)$ (locally stable) we know that also a 2−cycle exists on the axes, say C_{2A}, given by $\{(X_C^*, 0), (0, Y_C^*)\}$ which is locally a saddle. From the main property of the Cournot models (to have a separate second iterate function) the basin of attraction of the Cournot fixed point for the two-dimensional map $\widetilde{T}$ is given by the Cartesian product $\mathcal{B}_{\widetilde{T}}(C) = \mathcal{B}_F(X_C^*) \times \mathcal{B}_G(Y_C^*)$ where $\mathcal{B}_F(X_C^*)$ is the basin of attraction of the stable fixed point X_C^* for the map $F(X)$ and $\mathcal{B}_G(Y_C^*)$ is the basin of attraction of the stable fixed point Y_C^* for the map $G(Y) = g \circ f(Y)$. Moreover we also know that in continuous duopoly games the basins of attraction of the stable cycles are bounded by the lines belonging to the stable sets of the saddle cycles. But in our case the map is not continuous and we can see that in the positive phase plane (X, Y) we have no saddle cycle. It follows that the basins of attraction are separated by the lines associated with the discontinuities (see also Tramontana et al. (2008, 2010)). In the case shown in Fig.1a we have that $\mathcal{B}_F(X_C^*)$ is the whole segment $]0, 1[$ and $\mathcal{B}_G(y_C^*)$ is the whole segment $]0, d_Y[$. It follows that the basin of attraction of the Cournot fixed point for the two-dimensional map $\widetilde{T}$ is given by the Cartesian product $\mathcal{B}_{\widetilde{T}}(C) = \mathcal{B}_F(X_C^*) \times \mathcal{B}_G(Y_C^*) =]0, 1[\times]0, d_Y[$ as shown in Fig.1b. We can see that all the other points of the phase plane are mapped on the coordinate axes, either in the fixed point O or to converging to the 2−cycle saddle cycle C_{2A}.

This is not in contradiction with the fact that these cycles are locally unstable. From a dynamical point of view these cycles (the origin O and the saddle C_{2A}) are called stable *in weak sense* or *in Milnor sense* (see Milnor (1985)[2]). The reason why these cycles are stable in Milnor sense is the existence of "zero-branches" in the definition of the maps $F(X)$ and $G(Y)$. For the one-dimensional map $F(X)$ all the points in $]1, +\infty[$ are mapped into the origin, and thus also in the one-dimensional case (map $F(X)$) the basin of the origin is of positive measure. Similarly for $G(Y)$.

Contrarily to what occurs in continuous maps, we can see that in the case of discontinuous functions we can have multistability even if there are no saddle cycles, as further stable cycles may appear by BCB. In the case considered above in Fig.1, we can see that an increase in the parameter r leads to the appearance of a stable 2-cycle, as shown in Fig.2.

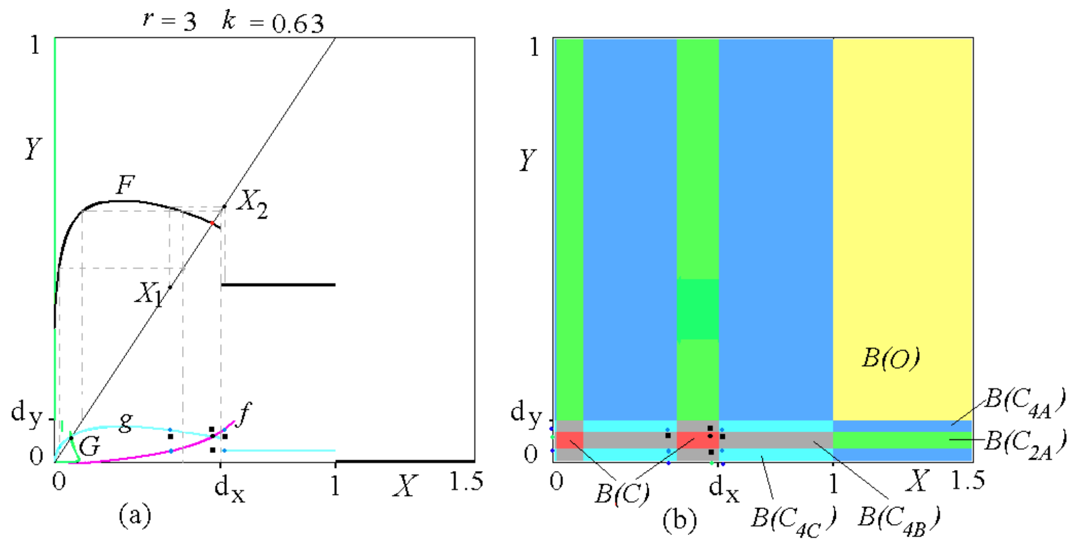

Fig.2 $r = 3$, $k = 0.63$. Reaction functions in (a). Basins of attraction in (b).

From the existence of a $2-$cycle of $F(X)$, say $\{X_1, X_2\}$, locally stable and coexisting with the stable Cournot fixed point, we can deduce the existence of two stable $4-$cycles (C_{4B}, C_{4C}) for our map $\widetilde{T}$, with periodic points all in the positive quadrant, and one more $4-$cycle (C_{4A}) with periodic points on the axes (locally a saddle). All the periodic points belong to the Cartesian product $\{0, X_1, X_C^*, X_2\} \times \{0, X_1, X_C^*, X_2\}$. In the case shown in Fig.2 the periodic points are evidenced. Correspondingly we can see that the related basins of attraction are modified (compare Fig.1b with Fig.2b). The rectangle which was previously the basin of

[2]a cycle is said stable in Milnor sense if it is locally unstable but its basin of attraction is of positive measure in the phase space.

the Cournot point now includes quite smaller rectangles as basin $\mathcal{B}_{\widetilde{T}}(C)$ plus two new basins $\mathcal{B}_{\widetilde{T}}(C_{4B})$ and $\mathcal{B}_{\widetilde{T}}(C_{4C})$. Similarly, the portion of phase plane previously the basin $\mathcal{B}_{\widetilde{T}}(C_{2A})$ now includes a smaller basin for $\mathcal{B}_{\widetilde{T}}(C_{2A})$ plus a new basin $\mathcal{B}_{\widetilde{T}}(C_{4A})$. For more details on the structure of the basins of attraction in discontinuous duopoly games we refer also to Tramontana et al. (2008). It is clear that as long as we have $g_C(X) < d_Y$ the function $F(X)$ is defined by $F_{CC}(X)$. The condition $g_C(X) < d_Y$ is satisfied for

$$x - \sqrt{x} + r\left(1 - \sqrt{k\frac{r}{4}}\right)^2 > 0 \tag{32}$$

and in its turn (32) is always satisfied if its minimum value is positive, which occurs if

$$r\sqrt{k} - 2\sqrt{r} + 1 > 0. \tag{33}$$

We can see that for $k > 1$ the inequality in (33) always holds, while for $k < 1$ it may be violated. Let

$$\sqrt{X_\pm} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4r(1 - \left(1 - \sqrt{k\frac{r}{4}}\right)^2} \tag{34}$$

$$= \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - (r\sqrt{k} - 2\sqrt{r})^2} \tag{35}$$

then for $X_- < X < X_+$ we have $F(X)$ defined by $F_{CS}(X)$. The equation from (33):

$$(BCB_{bas}) \; : \; r\sqrt{k} - 2\sqrt{r} + 1 = 0 \tag{36}$$

(that is $\sqrt{k} = \frac{2}{\sqrt{r}} - \frac{1}{r}$) or $\sqrt{r} = \frac{(1 \pm \sqrt{1-\sqrt{k}}\,)}{\sqrt{k}}$) corresponds to a global bifurcation whose effect is a change in the structure of the basins of attraction, as we can see comparing Fig.2 with Fig.3.

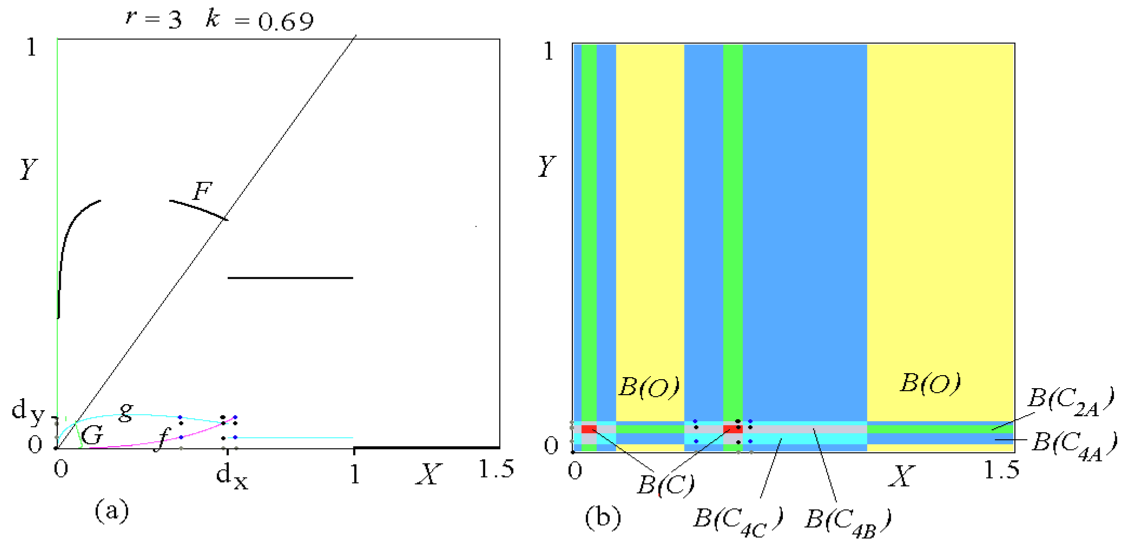

Fig.3 $r = 3$, $k = 0.69$. Reaction functions in (a). Basins of attraction in (b).

The BCB occurring when the condition in (36) holds (in our example increasing the parameter r), is associated with the appearance of new discontinuities in the function $F(X)$. In fact, after the bifurcation, for $X_- < X < X_+$ where $X_\pm$ are given in (34), the function $F(X)$ is constant (in the example shown in Fig.3a the value is high and not visible). So that the fate of such points in the iteration of the map are changed, and precisely now they end in the flat zero branch. Correspondingly the structure of the basins is changed, as we can see from Fig.3b, where the meaning of the colors is the same as in Fig.2b. There is a wider portion of points now ending in zero under iteration of the map $\widetilde{T}(X,Y)$. It is worth to notice that the rectangles and strips belonging to the basins are not finite in number, as it may appear from the numerical figures, because the continuous branch of $F(X)$ issuing from the origin implies infinitely many preimages having the origin as limit set. Thus the basins are formed by infinitely many rectangles or strips.

As we know, when the Cournot fixed point

$$C = (X_C^*, Y_C^*) = ((\frac{r}{1+r})^2, (\frac{1}{1+r})^2) \tag{37}$$

exists, it is stable as long as

$$3 - 2\sqrt{2} < r < 3 + 2\sqrt{2} \tag{38}$$

and a flip bifurcation of the fixed point of the function $F(X)$, corresponding to a degenerate Neimark-Sacker bifurcation of the Cournot fixed point, occurs at $r = 3 \pm 2\sqrt{2}$ (for details see also Tramontana et al.

(2010)). So, depending on the range, we may have a fixed point which, when existing, is always stable, or it may change stability in its region of existence. However the Cournot fixed point may not exist. A BCB leading to the appearance/disappearance of the Cournot equilibrium occurs when $d_X = X_C^*$, and similarly when $d_Y = Y_C^*$, that is for

$$(BCB_{X_C^*}) \ : \ k = \frac{4r}{(1+r)^2} \quad , \quad (BCB_{Y_C^*}) \ : \ k = \frac{4}{r(1+r)^2} \tag{39}$$

and a BCB leading to the appearance/disappearance of the Cournot equilibrium may also occur when the fixed point X_C^* merges with the discontinuity point X_+ given in (34), that is, $X_+ = X_C^*$ occurs for

$$\sqrt{1-(r\sqrt{k}-2\sqrt{r})^2} = \frac{r-1}{r+1} \tag{40}$$

leading to a bifurcation in the structure of the basins of attraction of the existing attractors. However, the related BCB curve, of equation

$$(BCB_C) \ : \ k = \frac{4}{r}(\frac{r+2}{r+1})^2 \tag{41}$$

seems to belong to the unfeasible region.

The curves $(k_1): \ k = 4r$ and $(k_2): \ k = \frac{4}{r}$ given in (28), bounding the feasible region, are shown in the bifurcation diagram on Fig.4, in the two-dimensional parameter plane (r, k). In the same figure also the BCB curve (BCB_{bas}) given in (36) is shown, as well as the BCB curves given in (39) and in (40), and the two lines $r = 3 \pm 2\sqrt{2}$ given in (38) bounding the analytic stability region of the Cournot fixed point (denoted S_C in Fig.4).

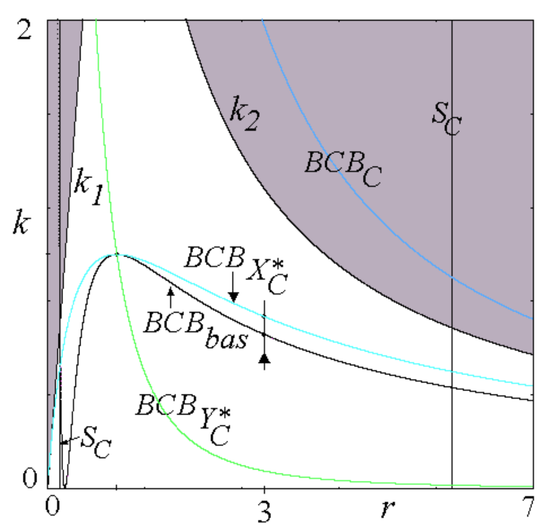

Fig.4 Bifurcation curves in the two dimensional parameter plane (r, k). The gray region is unfeasible.

The vertical line at $r = 3$ shown in Fig,4 includes the examples shown above, in Fig.s 1,2,3. As we can see, from $r = 0.63$ considered in Fig.2 to $r = 0.69$ considered in Fig.3 we have crossed the BCB curve given in (36). We recall again that the local eigenvalues are unchanged when varying the parameter k, but border collision bifurcations may occur. Indeed, increasing the parameter k the position of the points $X_{\pm}$ given in (34) changes, and a collision of X_{+} with the lowest periodic point (X_1) of the 2−cycle of $F(X)$ (which is associated with a periodic point of the 4−cycle C_{4A}, as well as an X−coordinate for the 4−cycles C_{4B} and C_{4C}), causes the disappearance of all the three 4−cycles for BCB, leaving a huge portion of points converging to zero. A very extreme example is shown in Fig.5 for $k = 0.7$.

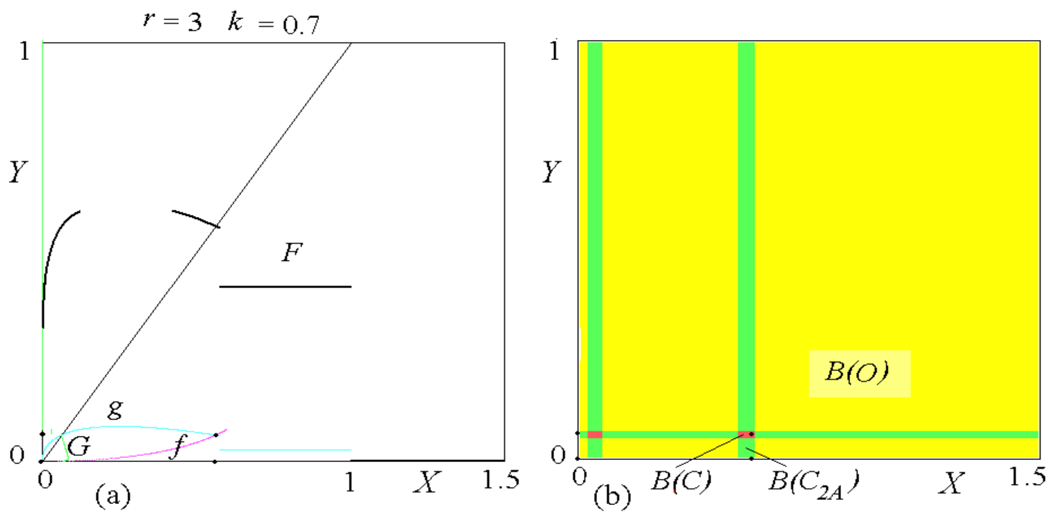

Fig.5 $r = 3$, $k = 0.7$. Reaction functions in (a). Basins of attraction in (b).

The Cournot point exists and is locally stable, but its basin of attraction is so small that may be considered really "unimportant" from the applied point of view (while the dominant state is zero, i.e. the origin, and some points converge to the saddle 2−cycle C_{2A} on the coordinate axes).

We have described above the disappearance of the 2−cycle for $F(X)$ (and thus of all the 4−cycles for the map $\widetilde{T}(X,Y)$) via BCB, but also its appearance occurs via BCB. Its analytical determination is more complicated due to the complex structure of the functions involved in our map. The periodic points $\{X_1, X_2\}$ of the 2−cycle of $F(X)$ are obtained solving for $F^2(X) = X$ (and discarding the fixed point of $F(X)$). In our case, we cannot find them explicitly. However, we can say that the appearance of the 2−cycle is associated with some BCB occurring when $X_1 = d_X$ or when $X_2 = 1$. A numerical investigation has been performed, reported in Fig.6. The colors there shown correspond to attracting cycles of different periods for the one-dimensional map $F(X)$, and also to cycles of the same period but of different nature. In fact, the attracting fixed point of the map $F(X)$ may be the Cournot fixed point C, or the Stackelberg fixed point S in the horizontal branch of the map, or the unstable origin O, and depending on the initial condition a point of the phase space may converge to one of them. The presence of two colors in Fig.6 denotes the coexistence. Besides the fixed points listed above, only a $2-$cycle for the map $F(X)$ has been detected numerically, in two regions, bounded by BCB curves (that is, this cycle is not associated with a flip bifurcation of the fixed point of $F(X)$). This means that the kind of cycles that we have described in the examples given in the previous figures, are all the possible cycles for the two-dimensional map $\widetilde{T}(X,Y)$.

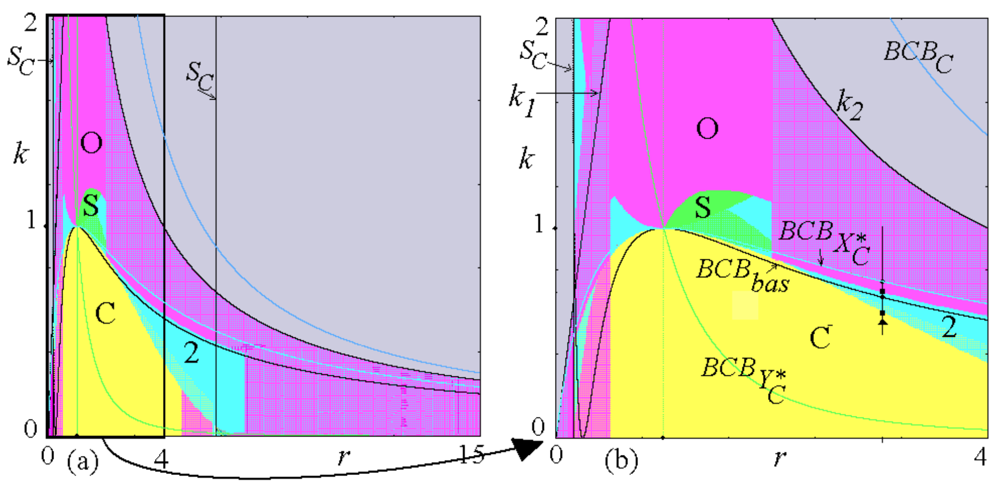

Fig.6 In (a): Bifurcation diagram in the two dimensional parameter plane (r,k). The gray region is unfeasible. The different colors refer to the three fixed points C, O, S and to the $2-$cycle of $F(X)$. In (b) an enlargement.

We can see that in Fig.6, as well as in Fig.4, the BCB curve given in (36) and the BCB curves given in (39) and (40), all intersect in the same point $(r,k)=(1,1)$. This is not surprising. We know that $r=1$ is a symmetry value, the stability of the Cournot point has the same behavior as r is increased or decreased. This is due to the fact that exchanging r with $\frac{1}{r}$ and X with Y, the model does not change, that is:

$$\widetilde{T}(X,Y,r)=\widetilde{T}(Y,X,\frac{1}{r}) \tag{42}$$

so it is enough to consider the bifurcations for $r>1$. Similar bifurcations also exist for $r<1$. In the particular case $r=1$ the Cournot fixed point becomes $C = (X_C^*, Y_C^*) = (\frac{1}{4}, \frac{1}{4})$ ant it is stable. The structure of the basins in the particular point $(r, k) = (1, 1)$ of the parameter plane is shown in Fig.7, the basin $\mathcal{B}_{\widetilde{T}}(C)$ is the square $\mathcal{B}_F(X_C^*) \times \mathcal{B}_G(Y_C^*) =]0, 1[\times]0, 1[$.

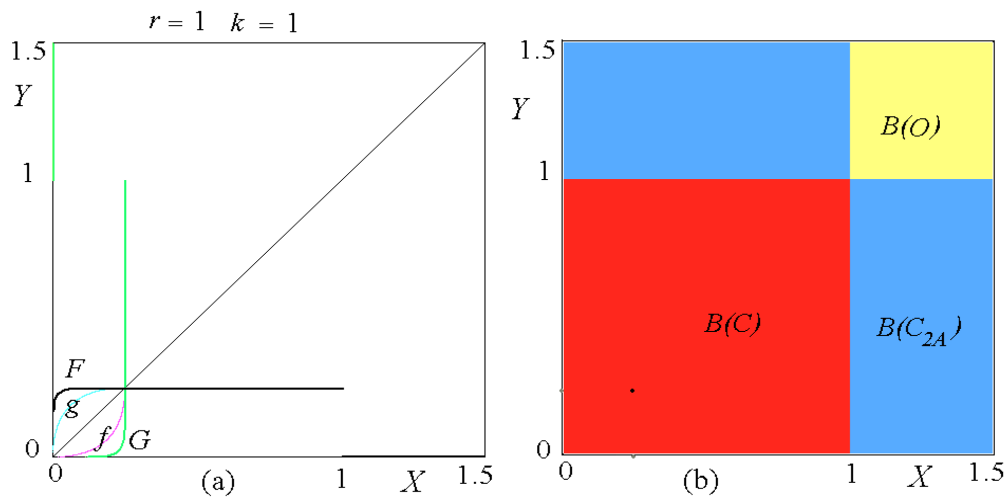

Fig.7 $r = 1$, $k = 1$. Reaction functions in (a). Basins of attraction in (b).

The dynamics in this case of basic model (with the zero branches), are associated with the cycles commented above. Fig.6 shows that only at low values of k $(k < 1)$ we can have a stable Cournot point, and when it becomes unstable or disappears by BCB, the dynamics are mainly associated with the coordinate axes, except for the small regions associated with a $2-$cycle of $F(X)$. That is, the dominating attracting sets are attractors in Milnor sense, belonging to the coordinate axes. It follows that the states are always alternating from $X = 0$ to $Y = 0$ and *vice versa*.

To improve the dynamic behavior, in the next Section we will modify the considered model.

4 The modified Cournot-Stackelberg model.

The undesired features of the basic model considered in the previous Section are associated with the fact that once that the state variable gets a zero value, it cannot be increased any longer, as $X = 0$ and $Y = 0$ are fixed points of the reaction curves (although unstable) and those axes are invariant and trapping in the phase-space. Thus a more suitable model, satisfying the intuitive behavior, is that a state variable X or Y can become very low, assuming a fixed low value, say ϵ, so that the last horizontal branch in the reaction functions is not zero but a constant value $\epsilon > 0$. Clearly this low value may also be different for the two state variables. However, for shake of simplicity, let us take the same low value for both. Thus the model we are now considering is the map T_ϵ, $T_\epsilon(x, y) = (x', y')$ defined as follows:

$$x' = f(y) = \begin{cases} f_C(y) = \sqrt{\frac{y}{a}} - y & if \;\; y \leq d_y = \frac{1}{a}\left(1 - \frac{1}{2}\sqrt{kr}\right)^2 \\ f_S(y) = \frac{b}{4a^2} & if \;\; d_y < y \leq \frac{1}{a} \\ \epsilon & if \;\;\;\;\;\;\;\; y > \frac{1}{a} \end{cases} \tag{43}$$

$$y' = g(x) = \begin{cases} g_C(x) = \sqrt{\frac{x}{b}} - x & if \;\; x \leq d_x = \frac{1}{b}\left(1 - \frac{1}{2}\sqrt{\frac{k}{r}}\right)^2 \\ g_S(x) = \frac{a}{4b^2} & if \;\; d_x < x \leq \frac{1}{b} \\ \epsilon & if \;\;\;\;\;\;\;\; x > \frac{1}{b} \end{cases} \tag{44}$$

Then, as before, by rescaling the variables, setting $X = bx$ and $Y = ay$, we obtain a two dimensional map which only depends on $(X, Y; r, k)$, $\widetilde{T}_\epsilon(X, Y) = (X', Y')$ (topologically conjugated with T_ϵ) given by:

$$X' = f(Y) = \begin{cases} f_C(Y) = r(\sqrt{Y} - Y) & if \;\; Y \leq d_Y = \left(1 - \frac{1}{2}\sqrt{kr}\right)^2 \\ f_S(Y) = \frac{r^2}{4} & if \;\; d_Y < Y \leq 1 \\ \epsilon & if \;\;\;\;\;\;\;\; Y > 1 \end{cases} \tag{45}$$

$$X' = g(X) = \begin{cases} g_C(X) = \frac{1}{r}(\sqrt{X} - X) & if \;\; X \leq d_X = \left(1 - \frac{1}{2}\sqrt{\frac{k}{r}}\right)^2 \\ g_S(X) = \frac{1}{4r^2} & if \;\; d_X < X \leq 1 \\ \epsilon & if \;\;\;\;\;\;\;\; X > 1 \end{cases} \tag{46}$$

It is clear that many results of the previous section hold also now. However, the main fact is that the zero state can no longer be reached. The function $F(X) = f(g(X))$ has now the origin which is a true repelling fixed point, while for $X > 1$ the function takes the constant value $F(X) = f(\epsilon)$, let us call this constant $X_m = f(\epsilon) = r(\sqrt{\epsilon} - \epsilon)$ which is the minimum value which can be reached by the iterated points of the map in the absorning interval. All the points X for which $g(X) > 1$ are mapped into ϵ, and thus $F(X)$ is mapped into X_m and the trajectory of this point converges to an attracting set of the map, which is a *superstable cycle* (as the derivative of the map in the periodic points higher than 1 is zero). Thus when the maximum of $F(X)$ exceeds 1 then ϵ (that is X_m for $F(X)$) is a periodic point of the map belonging to an attracting cycle. Stated in other words, once that the state variable X reaches a low value, the increasing branch of $F(X)$ issuing from the origin will push the state to increase again. It follows that instead of the fixed point O the trajectories are converging to a different cycle, with positive state variables. Clearly the period of the cycle depends on the value of ϵ and on the values of the other parameters.

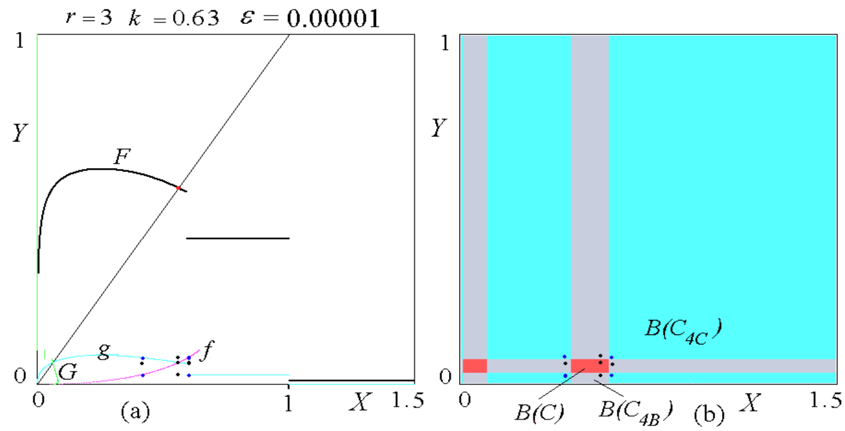

Fig.8 $r = 3$, $k = 0.63$, $\epsilon = 10^{-5}$. Reaction functions in (a). Basins of attraction in (b).

As an example, let us consider the case shown in Fig.2, with the new model assuming $\epsilon = 10^{-5}$: the existing cycles are exactly the same with the same coordinates. What are changed are the basins of attraction of the attracting cycles. The result is shown in Fig.8, where we can see that now the cycles on the coordinate axes are truly saddles, and no longer stable in Milnor sense: their stable set is a set of zero measure. The points in the phase plane that were converging to the axes in Fig.1b are now converging to the two stable cycles C_{4B} and C_{4C} for our map $\widetilde{T}_\epsilon$.

Similar changes occur for the case shown in Fig.3, now shown in Fig.9. As we can see, the bifurcation in the shape of $F(X)$ when we increase r crossing the BCB curve (BCB_{bas}) given in (36) is no longer important for the modified model, and the structure of the existing basins of attraction changes smoothly (that is, without a bifurcation).

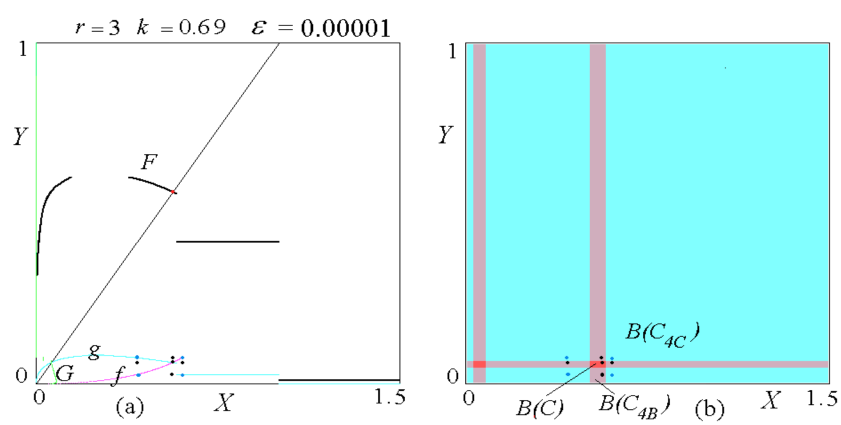

Fig.9 $r = 3$, $k = 0.69$, $\epsilon = 10^{-5}$. Reaction functions in (a). Basins of attraction in (b).

While a new situation occurs now after the BCB leading to the disappearance of the $2-$cycle of $F(X)$. As we have seen in the previous section the disappearance of the $2-$cycle leads to the dominance of the zero fixed point (see the example in Fig.5), while now, in the new map $F(X)$, the disappearance of the $2-$cycle via BCB leads to the appearance of a new attracting cycle with positive values, in the example a cycle of period 5 with points $\{X_i,\ i = 1, ..., 5\}$ (see Fig.10). For the map $\widetilde{T}_\epsilon$ this means the disappearance of the three $4-$cycles and the appearance of three cycles: one $5-$cycle C_{5A} together with three different $10-$cycles coexisting with the stable Cournot fixed point. All the periodic points belong to the Cartesian product $\{0,\ X_i,\ i = 1, ..., 5,\ X_C^*\} \times \{0,\ X_i,\ i = 1, ..., 5,\ X_C^*\}$. The shape of the reaction functions are shown in Fig.10a while in Fig.10b the five basins of attraction are illustrated.

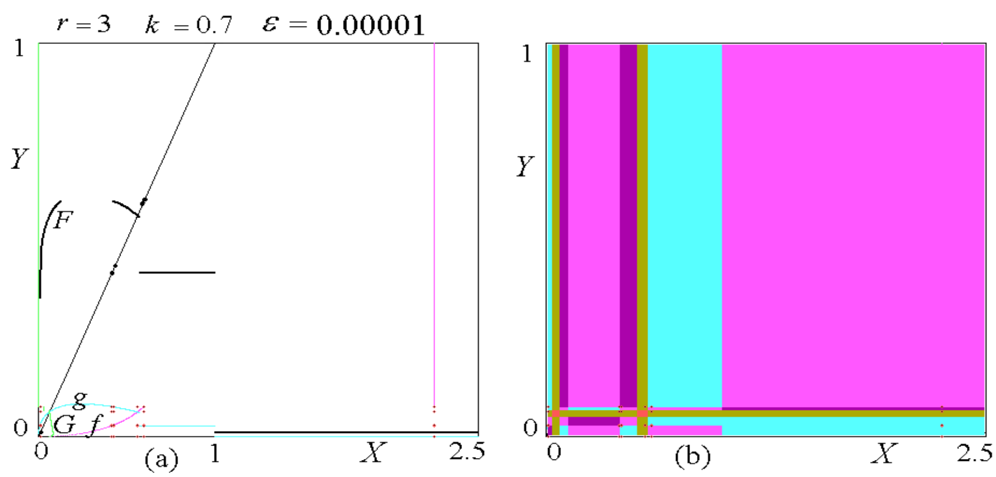

Fig.10 $r = 3$, $k = 0.7$, $\epsilon = 10^{-5}$. Reaction functions in (a). Basins of attraction in (b).

The main difference for the dynamics occurring in this modified model can be appreciated at low values of the parameter k. In fact, when the Cournot fixed point becomes unstable, other stable cycles exist which enrich very much the dynamics and the opportunities in the game. A numerical example is shown in Fig.11, where the two-dimensional bifurcation diagram in the parameter plane (r, k) is reported at fixed value $\epsilon = 10^{-5}$. We can see that on the right side of the line at which the Cournot changes its stability, several cycles can occur, as magnified in the two enlargements. Regions with different colors correspond to cycles of $F(X)$ with different periods. It is plain that similar regions also exist in the side with $r < 1$.

It is worth to notice that the period observed in the $X-$coordinate (corresponding to the period of the cycle for the map $F(X)$) may not correspond to the period of the cycle for the two-dimensional map because there may be points of the cycle with the same $X-$projection. For each cycle of the map $F(X)$ we have coexistence of many cycles for the two-dimensional game $\widetilde{T}_{\epsilon}$, following the rule already described in Bischi et al. (2008).

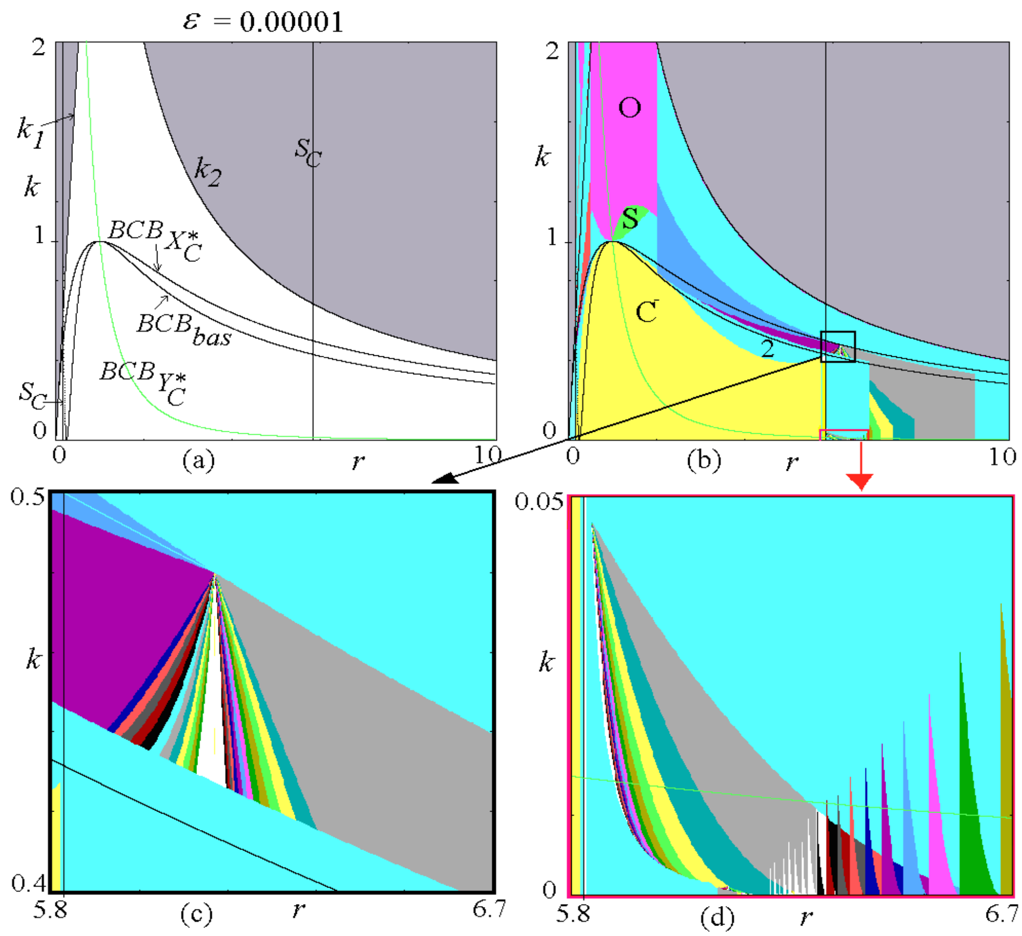

Fig.11 In (a): Bifurcation curves in the two dimensional parameter plane (r, k) at $\epsilon = 10^{-5}$ fixed. Colored bifurcation diagram in (b). Two enlargements are shown in (c) and (d).

The existence of cycles of any order may be better observed in the one-dimensional bifurcation diagram in which the state variable X of the map $F(X)$ is drawn as a function only of the parameter r, at fixed values of ϵ and k. An example is shown in Fig.12, where we have used $\epsilon = 10^{-5}$ and $k = 0.44$.

In the case shown in Fig.12 we can see that a sequence of BCBs occur. The period of the cycles of the map $F(X)$ is increasing with an increment of two units at some BCBs as the parameter r tends to a bifurcation value r^*, after which the period of the cycles decreases by two units at other BCBs. This sequence of bifurcations can be easily explained from the graph of the function $F(X)$. In Fig.13a we can see that for $r < r^*$ the value $X_m = f(\epsilon)$ is mapped below the Cournot fixed point, which is unstable, and thus a few iterations are needed to exit from that range reaching X_m again. As r tends to r^* the point $F(X_m)$ tends to the fixed point and the period increases because this periodic points must do more and more turns around the unstable Cournot point. Clearly the period tends to infinity and at the value $r = r^*$ we have $F(X_m) = X_C^*$ that is: the minimum value is preperiodic to the Cournot point. Then for $r > r^*$ we have $F(X_m) > x_C^*$ (as in the example shown in Fig.13b) and $F(X_m)$ increases, so that the period from very high tends to decrease.

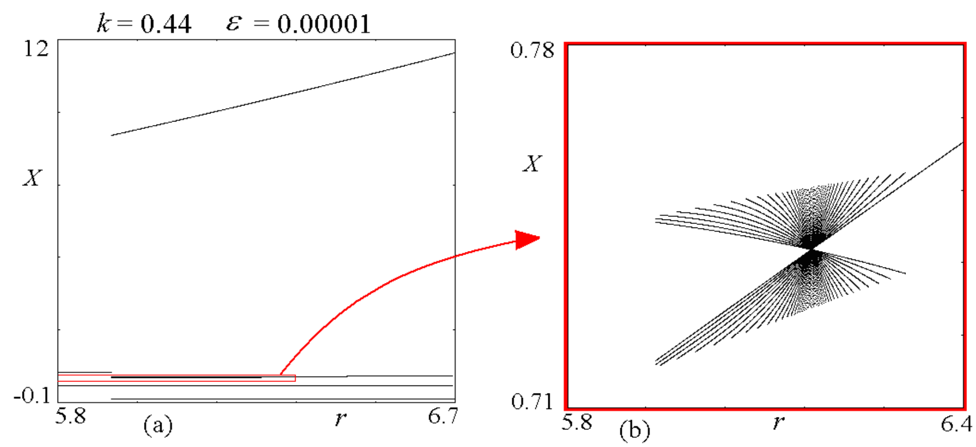

Fig.12 (a) One-dimensional bifurcation diagram of X as a function of the parameter r at $k = 0.44$ and $\epsilon = 10^{-5}$ fixed. In (b) an enlargement.

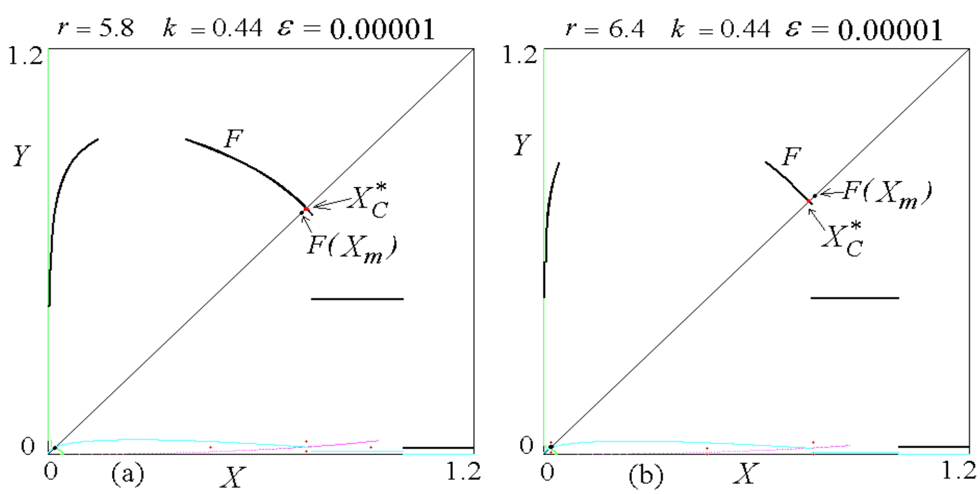

Fig.13 Reaction functions. In (a) at $r = 5.8$, $k = 0.44$, $\epsilon = 10^{-5}$ it is $F(X_m) < X_C^*$. In (b) at $r = 6.4$ it is $F(X_m) > X_C^*$.

We can also appreciate the dependence of the period of the cycles of the map (F and thus of $\widetilde{T}_\epsilon$) from the value of the constant ϵ. Fig.14a shows the parameter plane (r, ϵ) at fixed $k = 0.2$. The one-dimensional bifurcation diagrams at $r = 8$ fixed as a function of ϵ is shown in Fig.14b, evidencing the same phenomenon explained above: from a $2-$cycle, the period of the cycles tents to infinity and then decrease up to the period one of the fixed point S (Stackelberg).

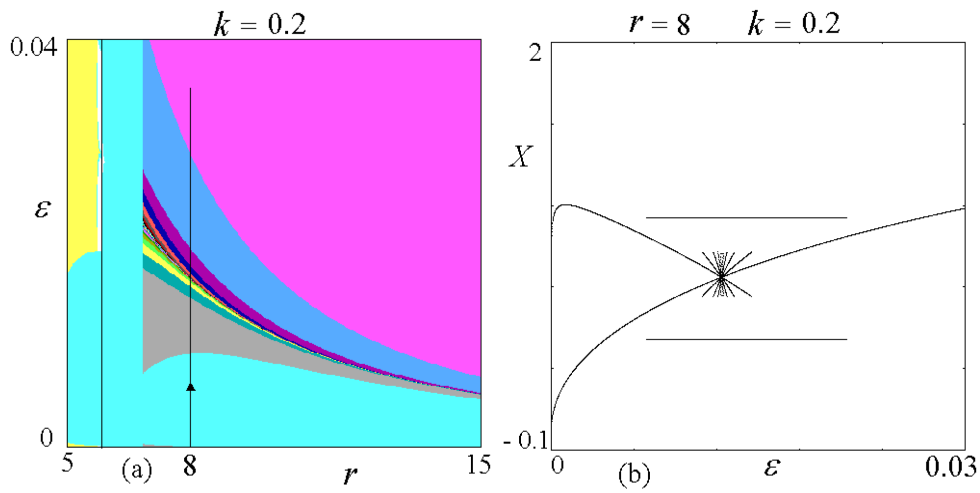

Fig.14 In (a): Bifurcation diagram in the two dimensional parameter plane (r, ϵ) at $k = 0.2$ fixed. In (b) one-dimensional bifurcation diagram of X as a function of the parameter ϵ at $r = 0.8$ and $k = 0.2$ fixed.

5 Conclusions

A new Cournot-Stackelberg duopoly model has been here investigated with respect to its global properties. As we have shown in Section 3, the main result with respect to Cournot models already known in the literature (which can also have complex dynamics), is that the Cournot-Stackelberg duopoly model here considered has always coexisting stable cycles of low periods. The matter is which points of the space phase leads to a stable equilibrium or to a cycle. The structure of the basins of attraction shows that many points are attracted to the coordinate axes, which means periodically zero production. This undesired effect is however due to a zero branch in the reaction functions, which is reasonable and realistic to substitute with any small positive quantity. The result, shown in Section 4, is that the modified model is always stable and with cycles having periodic positive quantitities. However, now the period may be any integer number, depending on the parameters and on the small constant value assumed in the production.

References

Ahmed, E., Agiza, H.N., Hassan, S.Z., 2000. On modifications of Puu's dynamical duopoly. Chaos, Solitons & Fractals 11, 1025-1028.

Agiza, H.N., Elsadany, A.A., 2004. Chaotic dynamics in nonlinear duopoly game with heterogeneous players. Applied Mathematics & Computation 149, 843-860.

Agliari, A., 2006. Homoclinic connections and subcritical Neimark bifurcations in a duopoly model with adaptively adjusted productions. Chaos Solitons & Fractals 29, 739-755.

Agliari, A., Gardini, L., Puu, T., 2006a. Global bifurcation in duopoly when the fixed point is destabilized via a subcritical Neimark bifurcation. International Game Theory Review 8, 1-20.

Agliari, A., Chiarella, C., Gardini, L., 2006b. A Re-evaluation of the Adaptive Expectations in Light of Global Nonlinear Dynamic Analysis. Journal of Economic Behavior and Organization 60, 526-552.

Angelini, N., Dieci, R., Nardini, F., 2009. Bifurcation analysis of a dynamic duopoly model with heterogeneous costs and behavioural rules. Mathematics and Computers in Simulation 79, 3179-3196.

Bischi, G.I., Mammana, C., Gardini, L., 2000. Multistability and cyclic attractors in duopoly games. Chaos Solitons & Fractals 11, 543-564.

Bischi, G.I., Chiarella, C., Kopel, M., Szidarovszky, F., 2009, Nonlinear Oligopolies: Stability and Bifurcations. Springer-Verlag, New York.

Cournot, A., 1838. Récherces sur les principes mathématiques de la théorie des richesses, Dunod, Paris.

Kopel, M., 1996. Simple and complex adjustment dynamics in Cournot duopoly models. Chaos, Solitons & Fractals 7, 2031-2048.

Milnor, J., 1985. On the concept of attractor. Communications in Mathematical Physics 99, 177-195.

Puu, T., 1991. Chaos in duopoly pricing. Chaos, Solitons & Fractals 1, 573-581.

Puu, T., 1998. The chaotic duopolists revisited. Journal of Economic behavior & Organization 33, 385-394.

Puu, T., Sushko, I. (Ed.s), 2002. Oligopoly and Complex Dynamics: Models and Tools. Springer, New York.

Puu, T. 2009. Unifying Cournot and Stackeberg duopoly in one dynamical system, in C. Chiarella, G.I. Bischi, L. Gardini (Eds.), Nonlinear Dynamics in Economics and Finace, Springer-Verlag, Berlin 2009.

Puu, T., 2010. Dynamics of Stackelberg Duopoly, in T. Puu and A. Panchuk (Eds.), Advances in Nonlinear Economic Dynamics, Nova Science Inc., New York, 2010.

von Stackelberg, H., Marktform und Gleichgewicht, Julius Springer, Berlin, 1934.

Tramontana, F., Gardini, L., Puu, T., 2008. Duopoly games with alternative technologies. Journal of Economic Dynamics & Control 33, 250-265.

Tramontana, F., 2010. Heterogeneous duopoly with isoelastic demand function. Economic Modelling 27, 350-357.

Tramontana, F., Gardini, L., Puu, T., 2010. New properties of the Cournot duopoly with isoelastic demand and constant unit costs (submitted for publication).