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"Controlling Chaos Through Local Knowledge"

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WP-EMS # 2008/10

Controlling Chaos Through Local Knowledge

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Abstract.

We propose an duopoly game where quantity-setting firms have incomplete information about the demand function. In each time step, they solve a profit maximization problem assuming a linear local approximation of the demand function. In particular, we construct an example using the well known duopoly Puu's model with isoelastic demand function and constant marginal costs. An explicit form of the dynamical system that describes the time evolution of the duopoly game with boundedly rational players is given. The main result is the global stability of the system.

JEL Classification: L13; D83; C61; C62

1 Introduction

An oligopoly is a market structure between the two extreme cases of monopoly and of perfect competition; from a theoretical point of view, Cournot, in 1838 [2] has introduced the first formal theory of oligopoly and the duopolistic Cournot model is one of the most well know subjects of economic dynamics.

Often, in a static setting, the model is solved using the notion of Nash equilibrium, this kind of solution in market games implies that each firms knows the entire demand curve of the good it produces, has perfect foresight about the next period production of the other firms operating in the same market; from a computational point of view, Nash equilibrium implies also that the players are able to solve a one period optimization problem.

In 1978 Rand [7] and Poston and Steward [4], in a dynamic setting based on the reaction functions, showed that, under suitable conditions, simple duopoly would lead to complex dynamic phenomena. The dynamical complexity arises from the unimodal character of the reaction function, i.e., the curve that shows how to react, in the optimal way, with respect to the past strategy of the competitor. In the last two paper, the reaction functions are not derived from the solution of an optimization problem but are proposed by the authors in an exogenous and abstract way.

In 1991 Puu [5] suggested the first example of duopolistic model with unimodal reaction functions, derived from the optimization of profit functions, that gives rise to complex phenomena. It is assumed an isoelastic demand function, which always arises when the consumer have Cobb–Douglas preferences type, combined with constant marginal costs. The model was shown to produce a period doubling sequence of flip bifurcations ending in chaos for the outputs of each of the two competitors.

Since then, many contributions has been proposed in order to enrich the simple economic assumptions of the Cournot original model. Among numerous references listed in [6] and in [8], the reader can find many different examples of the Cournot oligopoly game with complex dynamics.

This paper investigates a learning duopoly model; the oligopolists do not know the demand function they face, but they believe it is linear and they estimate such a linear function through the knowledge of the current market state in terms of quantity and price and on the basis of local knowledge of the demand curve. This adjustment process was introduced by Tuinstra [9], and Bischi et al. [1], Naimzada and Sbragia [3] who analyzed it for price-setting and quantity-setting oligopolies, respectively.

The main purpose of this paper is to investigate the dynamic behavior of a duopoly game with two firms learning about the demand curve through the local linear approximation process; to this end, we will use Puu's model and demonstrate our main result; the discrete dynamical system that arises considering two boundedly rational players that use local linear approximation, is *globally stable*. The plan of the paper is as follows: In section 2, is constructed and described Puu's simple nonlinear duopoly model . In Section 3, is presented a duopoly model with isoelastic demand curve and constant marginal costs and local linear approximation of the demand curve. In section 4, the dynamics of a duopoly game with boundedly rational player is analyzed. Section 5 considers an example in which is analyzed numerically and graphically the evolution of the quantities chosen by the oligopolists. In section 6 we compare the expected profits of the firms in the two different settings, the one with global knowledge and the one with local knowledge.

2 The Puu model

Puu [5] builds a model using a demand function in which price (P) is reciprocal to the total demand function (Q):

$$P = \frac{1}{Q} \tag{1}$$

The (1) represents an *isoelastic* demand function, because the elasticity of the demand is constant. In the duopoly case there are two firms (firm 1 and firm 2) producing two perfect substitute goods (q_1 and q_2). Under the assumption that total demand is equal to total supply, we know that $Q = q_1 + q_2$.

Analyzing the nonlinear Cournot output adjustment, we know that at period $t+1$, each firm produces a quantity of good that depends on the expected price for the period, which depends on the quantities produced by both the duopolists. So, each firm has to make an expectation on the competitor's production. In this way firm 1 produces a quantity $q_1(t+1)$, given the expected production of firm 2 ($q_2^e(t+1)$), whereas firm 2 produces a quantity $q_2(t+1)$, given the expected production of firm 1 ($q_1^e(t+1)$). The chosen quantities $q_1(t+1)$ and $q_2(t+1)$ are such that firms maximize their expected profits that are (under the assumption of linear costs):

$$\begin{aligned} \pi_1^e(t+1) &= \frac{q_1(t+1)}{q_1(t+1)+q_2^e(t+1)} - c_1 q_1(t+1) \\ \pi_2^e(t+1) &= \frac{q_2(t+1)}{q_2(t+1)+q_1^e(t+1)} - c_2 q_2(t+1) \end{aligned} \tag{2}$$

where c_1 and c_2 are the marginal costs of the firms 1 and 2 respectively.

Maximizing profits using *naïve* expectations (i.e. $q_1^e(t+1) = q_1(t)$ and $q_2^e(t+1) = q_2(t)$) we obtain the dynamic process:

$$\begin{cases} q_1(t+1) = \sqrt{\frac{q_2(t)}{c_1}} - q_2(t) \\ q_2(t+1) = \sqrt{\frac{q_1(t)}{c_2}} - q_1(t) \end{cases} \tag{3}$$

where productions are both positive if $\frac{c_2}{c_1} \in \left[\frac{4}{25}, \frac{25}{4}\right]$.

The system (3) has two fixed points: a trivial one ($q_1 = q_2 = 0$) and a nontrivial fixed point given by:

$$q_1^c = \frac{c_2}{(c_1+c_2)^2} \quad ; \quad q_2^c = \frac{c_1}{(c_1+c_2)^2} \tag{4}$$

that we can call *Cournot point*. In the Cournot point the ratios of the outputs and the profits are given by:

$$\frac{q_1^c}{q_2^c} = \frac{c_2}{c_1} \quad ; \quad \frac{\pi_1^c}{\pi_2^c} = \left(\frac{c_2}{c_1}\right)^2 \tag{5}$$

The results (5) permit to conclude that (under the assumptions of isoelastic demand function, linear costs and naïve expectations):

Theorem 1 *At the Cournot point the more efficient firm obtain an higher profit and produces more output than the other one*

In the theorem the terms "more efficient" indicate the firm whose marginal cost is lower, in fact:

$$\begin{array}{l} c_1 \leq c_2 \text{ (resp. } c_1 > c_2) \Rightarrow q_1^c \geq q_2^c \text{ (resp. } q_1^c < q_2^c) \\ c_1 \leq c_2 \text{ (resp. } c_1 > c_2) \Rightarrow \pi_1^c \geq \pi_2^c \text{ (resp. } \pi_1^c < \pi_2^c) \end{array} \tag{6}$$

In order to analyze the local stability of the Cournot point, we evaluate in that point the Jacobian matrix:

$$J_c = \begin{bmatrix} 0 & \frac{(c_2-c_1)}{2c_1} \\ \frac{(c_1-c_2)}{2c_2} & 0 \end{bmatrix} \tag{7}$$

Trace and determinant of J_c ($Tr(J_c)$ and $Det(J_c)$) are 1 and $(c_1-c_2)^2/4c_1c_2$ respectively. The eigenvalues ($\xi_{1,2}$) are the solutions of the *characteristic equation* $\xi^2 - Tr(J_c)\xi + Det(J_c)$ and the Cournot point is stable if the absolute value of both the eigenvalues are lower than one. This condition is realized when:

$$(3-2\sqrt{2}) < \frac{c_2}{c_1} < (3+2\sqrt{2}) \tag{8}$$

The condition (8) implies that in the special case of $c_1 = c_2$ the Cournot point is stable. When the condition (8) is violated, the Cournot point is unstable and increasing the ratio c_2/c_1 we can observe the period doubling sequence of bifurcations leading to chaos.

3 The conjectured demand function

In this section we propose the boundedly rational adjustment process in which we will analyze the evolution of the quantities chosen by the oligopolists. Firms produce homogeneous products and the price depends on the total output of the industry according to the inverse demand function:

$$p(t) = f(Q(t)) \tag{9}$$

where $Q(t) = \sum_{j=1}^{n} q_j$ is the total output of the oligopoly market and n is the number of the oligopolistic firms operating in the market.

The inverse demand function satisfies the canonical conditions about demand: $f(Q(t)) > 0$ and $f'(Q(t) \leq 0$, with $Q > 0$.

The total costs functions are linear: $TC_i = c_i q_i$ with $c_i > 0$ and $i = 1, ..., n$ and where c_i represents the marginal cost of the i-th firm.

In time period t each firm through the local knowledge of the demand function defines the conjectured demand function and then the firm will optimize the expected profits.

The local knowledge of the demand function means that the firm knows the current market price, $p(t)$, the corresponding quantity $Q(t)$, produced by the firms and demanded by the market and, through market experiments, the local linear approximation of the demand function in the point represented by the current price and by the current quantity produced, $(p(t), Q(t))$. These elements are sufficient for the local approximation of the demand function; we can define the conjectured demand for the next period $t + 1$:

$$p_i^e(t+1) = p(t) + f'(Q(t))(Q^e(t+1) - Q(t)) \tag{10}$$

where $Q^e(t+1)$ represents the aggregate conjectured production for time $t+1$. Then using the demand function we obtain:

$$p_i^e(t+1) = f(Q(t)) + f'(Q(t))(Q^e(t+1) - Q(t)) \tag{11}$$

We suppose that all the players adopt the cournotian expectations formation hypotesis, i.e. the static expectations; let $q_{-i}\ (t)$ defines the production at time t of the firms other than i,

$q_{-i}(t) = \sum_{j=1}^{n} q_j(t)$ with $j \neq i$, then the conjectured demand function becomes:

$$p_i^e(t+1) = f(Q(t)) + f'(Q(t))(q_i(t+1) + q_{-i}^e(t+1) - Q(t)) \tag{12}$$

where $q_{-i}^e(t+1)$ is the expectation by the i-th firm about the production in period $t+1$ by the other firms. According to cournotian hypotesis of static expectation firm i expects that other firms will produce the quantity of the previous period: $q_{-i}^e(t+1) = q_{-i}(t) = \displaystyle\sum_{j=1}^{n} q_j(t)$ with $j \neq i$, then eq.(12) becomes:

$$p_i^e(t+1) = f(Q(t)) + f'(Q(t))(q_i(t+1) + \sum_{j=1, j\neq i}^{n} q_j(t) - \sum_{j=1}^{n} q_j(t)) \tag{13}$$

Finally we have:

$$p_i^e(t+1) = f(Q(t)) + f'(Q(t))(q_i(t+1) - q_i(t)) \tag{14}$$

In fig.1 we represent the market conditions, $(Q(t), p(t))$ and the outcome of the learning process:

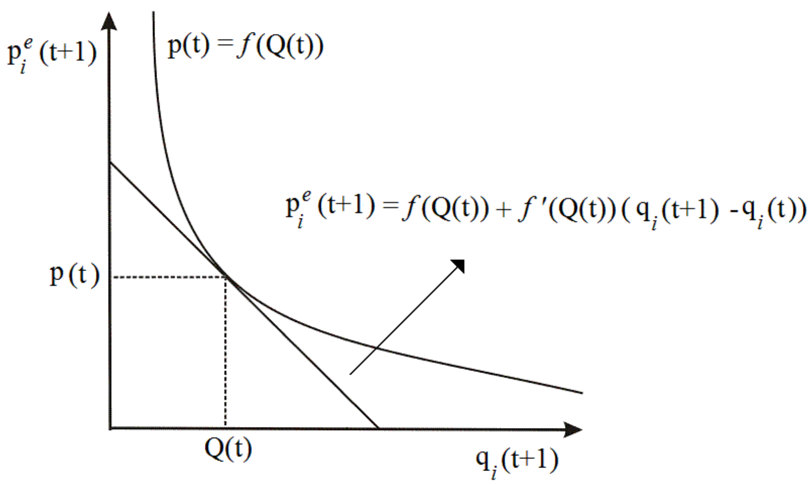

Figure 1. Conjecured demand function

The producers are quantity setting, so at each time period t they decide the next period production $q_i(t+1)$ by maximizing the expected profit at period $t+1$, $\pi_i^e(t+1)$:

$$q_i(t+1) = \arg\max_{q_i(t+1)} \pi_i^e(t+1) = \arg\max_{q_i(t+1)} [p_i^e(t+1)q_i(t+1) - c_i q_i(t+1)] \tag{15}$$

The first order condition for the maximization of the i-th expected profit is:

$$\frac{\partial \pi_i^e(t+1)}{\partial q_i(t+1)} = f(Q(t)) + 2q_i(t+1)f'(Q(t) - q_i(t)f'(Q(t)) - c_i = 0 \tag{16}$$

It is easily verified the second order condition.

If the dynamics of the system are based upon the best response function, the evolution of the quantities produced by the firms is defined by the following system of first order nonlinear difference equations:

$$\begin{aligned} q_1(t+1) &= \frac{q_1(t)}{2} + \frac{c_1 - f(Q(t)}{2f'(Q(t))} \\ q_2(t+1) &= \frac{q_2(t)}{2} + \frac{c_2 - f(Q(t)}{2f'(Q(t))} \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ q_n(t+1) &= \frac{q_n(t)}{2} + \frac{c_n - f(Q(t)}{2f'(Q(t))} \end{aligned} \tag{17}$$

4 Dynamic analysis of the Puu model with boundedly rational agents

In this section we consider a duopoly model in which the demand function is the isoelastic one. The model (17) with $n = 2$ and demand function (1) becomes a two dimensional dynamical system, defined by the iterated map:

$$T : \begin{cases} q_1(t+1) = \frac{q_1(t)}{2} + \left[\frac{1-(q_1(t)+q_2(t))c_1}{2}\right](q_1(t) + q_2(t)) \\ q_2(t+1) = \frac{q_2(t)}{2} + \left[\frac{1-(q_1(t)+q_2(t))c_2}{2}\right](q_1(t) + q_2(t)) \end{cases} \tag{18}$$

The relationship between the stationary equilibrium of the map (18) and the Nash equilibrium of the corresponding static duopoly game is defined by the following proposition given by Bischi et al. [1]:

Proposition 1 *The dynamical system (18) has an unique equilibrium, given by $q^* = (q_1^*, q_2^*)$, with*

$$\begin{aligned} q_1^* &= \frac{c_2}{(c_1+c_2)^2} \\ q_2^* &= \frac{c_1}{(c_1+c_2)^2} \end{aligned} \tag{19}$$

According to Propos.1 the stationary state q^* is also the Nash equilibrium of the Cournot duopoly game with isoelastic demand function. We can note that the steady state (19) is the same of the Puu duopoly model (4).

The main result of our work concerns the global stability of the Nash equilibrum:

Proposition 2 *The dynamical system (18) is globally asymptotically stable.*

Proof Let us introduce the auxiliary variable $S(t)$ defined as the aggregate production at each time period t: $S(t) = q_1(t) + q_2(t)$. Now, summing up the left hand sides and the right hand sides of the equations forming the system (18) we obtain the three-dimensional system:

$$T_1 : \begin{cases} q_1(t+1) = \dfrac{q_1(t)}{2} + \left[\dfrac{1 - S(t)c_1}{2}\right] S(t) \\ q_2(t+1) = \dfrac{q_2(t)}{2} + \left[\dfrac{1 - S(t)c_2}{2}\right] S(t) \\ S(t+1) = \dfrac{3}{2} S(t) - \dfrac{(c_1 + c_2)}{2} S^2(t) \end{cases} \tag{20}$$

The third equation of the system (20) implies that the aggregate production of a period only depends on its value on the previous period. In particular, the right hand side of the third equation represents a concave parabola with two fixed points: the trivial one $S_0^* = 0$ (always unstable) and another one characterized by a positive value of the aggregate production $S_1^* = \frac{1}{(c_1+c_2)} = q_1^* + q_2^*$ (Fig.2).

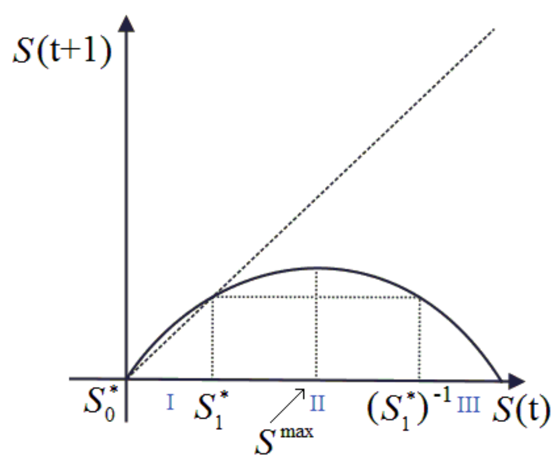

Figure 2. The aggregate production

The positive steady state S_1^* is always located on the increasing branch of the parabola (i.e. S_1^* is lower than the maximum point $S^{\max} = \frac{3}{2(c_1+c_2)}$) and this implies that it is globally asymptotically stable. In fact, if the initial condition is located on the region I ($0 < S < S_1^*$) the function is located above the diagonal, where $S(t+1) > S(t)$, and the succession of the values of the aggregated production is bounded by S_1^*.The monotonicity and the boundedness are sufficient conditions to conclude that starting from I the sequence of aggregate production converge monotonically to S_1^*. If S_0 is located inside the region III ($S > (S_1^*)^{-1}$) then in period 1 the aggregate production will be a value belonging to the region I and from there we have seen that the convergence is monotonic to the positive steady state. For the region II the mechanism is analogue to the one used for the region I, with the only difference that the monotonic convergence is decreasing.

This implies that the line $S = S_1^*$, which is mapped into itself by T_1, is globally attracting for the trajectories of T_1. In other words, the limit set of any trajectory of the map T_1 belongs to the trapping line $S = S_1^*$ (*line of ω-limit set*) and is an invariant set of the restriction of T_1 to such line, which can be identified with the two one-dimensional linear maps (*limiting maps*):

$$\begin{aligned} T_{\omega 1} : q_1(t+1) &= \frac{q_1(t)}{2} + \left[\frac{1 - S_1^* c_1}{2}\right] S_1^* \\ T_{\omega 2} : q_2(t+1) &= \frac{q_2(t)}{2} + \left[\frac{1 - S_2^* c_2}{2}\right] S_2^* \end{aligned} \tag{21}$$

The steady states of the limit maps are q_1^* and q_2^* respectively and they are both globally asymptotically stable, in fact the lines on the right hand side of $T_{\omega 1}$ and $T_{\omega 2}$ have both a slope equal to 0.5.■

This result implies that, differently from the Puu's model, even if the difference between the marginal costs of the firms is quite marked, their productions will always converge to the Nash equilibrium, i.e. a lower degree of rationality increases the stability of the system. In particular, the production at each time period is given by the following proposition:

Proposition 3 *The trajectory in each period given by the map (18) is defined by:*

$$\begin{aligned} q_1(t) &= \left(\tfrac{1}{2}\right)^t q_1(0) + \sum_{k=0}^{t-1} \left(\tfrac{1}{2}\right)^{t-k-1} z_1(k) \\ q_2(t) &= \left(\tfrac{1}{2}\right)^t q_2(0) + \sum_{k=0}^{t-1} \left(\tfrac{1}{2}\right)^{t-k-1} z_2(k) \end{aligned} \tag{22}$$

where $q_1(0)$ and $q_2(0)$ are the production in the initial period and $z_s(k) = S(k)(1 - S(k)c_s)/2$ for $s = 1, 2$.

Proof Using again the aggregate production variable $S(t)$ we can rewrite the map (18) in the following manner:

$$\begin{cases} q_1(t+1) = \frac{q_1(t)}{2} + \left[\frac{1-S(t)c_1}{2}\right] S(t) \\ q_2(t+1) = \frac{q_2(t)}{2} + \left[\frac{1-S(t)c_2}{2}\right] S(t) \end{cases} \tag{23}$$

If we introduce a new variable $z_s(t) \equiv S(t)(1 - S(t)c_s)/2$ the system becomes:

$$\begin{cases} q_1(t+1) = \frac{q_1(t)}{2} + z_1(t) \\ q_2(t+1) = \frac{q_2(t)}{2} + z_2(t) \end{cases} \tag{24}$$

which is a system formed by a couple of non-autonomous first-order linear difference equations of the form: $q_s(t+1) = aq_s(t) + z_s(t)$,with $a = 1/2$ and $s = 1, 2$ and from which the trajectories (22) derive.■

5 The dynamics of quantities and profit functions: an example

In this section we propose an example in which we analyze the evolution of the quantities chosen by the oligopolists. We are also going to show how the firms build every period an approximation of the demand function and an approximation of the profit function. With our numerical exercise we intend to show, step by step, how local approximation of the demand function operates in the particular case of a Cournot duopoly game ($n = 2$) with an isoelastic demand function:

$$p(t) = f(q_1(t) + q_2(t)) = \frac{1}{q_1(t) + q_2(t)} \tag{25}$$

together with linear cost functions:

$$C_i(t) = c_i q_i, \quad i = 1, 2 \tag{26}$$

we can substitute the demand function (25) and the cost function (26) into (18) to obtain the reaction functions.

Let us consider the case in which producers have different marginal costs, in particular: $c_1 = 0.8$ and $c_2 = 0.6$. The Nash equilibrium, according to (19), is $q^* \simeq [0.3061; 0.4081]$, so the equilibrium quantity of the firm 1 is lower than the competitor one, and this is a consequence of the difference between their marginal costs ($c_1 > c_2$). This case clearly violates the condition (8) for the stability of the Nash equilibrium in the Puu model. Proposition 2 tells us that this is not the case in our model in which the equilibrium is globally asymptotically stable. Let us compare the two different dynamics.

Suppose that at the initial period ($t = 0$) we have:

$$q_1(0) = 0.4 \ > q_1^* \ \ ; \quad q_2(0) = 0.1 < q_2^* \tag{27}$$

so firm 1 produces more than its Nash equilibrium quantity, whereas firm 2 produces less than it. We can find the price using the demand function (25):

$$p(0) = \frac{1}{0.4 + 0.1} = 2 \tag{28}$$

At the end of the initial period, the firms make their expectations on the next period ($t = 1$).

Let us consider firm 1. Its real profit depends on the quantities produced by both the competitors:

$$\pi_1(t) = p(t)q_1(t) - c_1 q_1(t) = \frac{q_1(t)}{q_1(t) + q_2(t)} - c_1 q_1(t) \tag{29}$$

in particular, in the initial period, we can substitute the value of the initial choice of the competitor obtaining the profit of the firm 1 as a function of its initial choice:

$$\pi_1|_{q_2 = q_2(0)} = \frac{q_1}{q_1 + 0.1} - 0.8q_1 \tag{30}$$

We can see that $q_1 = q_1(0) = 0.4$ does not maximize (30) but this is not surprising because the initial conditions are not the results of a decision process of the firms.

At the end of the period firm 1 knows the value of the partial derivative of the demand function in the point corresponding to the total output produced in the initial period:

$$f_1(t) = \frac{\partial f(q_1(t) + q_2(t))}{\partial q_1(t)} = -\frac{1}{(q_1(t) + q_2(t))^2} \Longrightarrow f_1(0) = -\frac{1}{(q_1(0) + q_2(0))^2} = -4 \tag{31}$$

which can be used to compute the expected price for the period 1 according to (10):

$$p_1^e(1) = p(0) + f_1(0)(q_1 - q_1(0)) = 2 - 4(q_1 - 0.4) = 3.6 - 4q_1 \tag{32}$$

so the price expected by firm 1 only depends on its own quantity and does not depend on the expectation concerning the competitor's quantity. Firm 1 can use (32) to obtain its profit for the period 1:

$$\pi_1^e(1) = p_1^e(1)q_1 - c_1 q_1 = -4q_1^2 + 2.8q_1 \tag{33}$$

The expected profit function (33) has a maximum in the point $q_1 = 0.35$ which is the quantity chosen by the firm 1 for the period 1 ($q_1(1)$). The real profit obtained by the firm 1 at the time $t = 1$, depends on the quantity chosen by the concurrent firm which, following the same mechanism seen for firm 1, is $q_2(1) = 0.225$:

$$\pi_1|_{q_2 = q_2(1)} = \frac{q_1}{q_1 + 0.225} - 0.8q_1 \tag{34}$$

At the end of the period 1, the firm correctly estimates the partial derivative of the demand function:

$$f_1(1) = -\frac{1}{(q_1(1) + q_2(1))^2} = -3.0246 \tag{35}$$

and use it to forecasts the price for the next period ($t = 2$):

$$p_1^e(2) = p(1) + f_1(1)(q_1 - q_1(0)) = 1.739 - 3.0246(q_1 - 0.35) = 2.79761 - 3.0246q_1 \tag{36}$$

which gives rise to this expected profit function:

$$\pi_1^e(2) = p_1^e(2)q_1 - c_1q_1 = -3.0246q_1^2 + 1.99761q_1 \tag{37}$$

Now, in order to calculate the real profit obtained by the firm 1, we need to know the quantity chosen by the concurrent firm, $q_2(2) \simeq 0.3$, and use it to obtain the right section of the real profit function:

$$\pi_1|_{q_2=q_2(2)} = \frac{q_1}{q_1 + 0.3} - 0.8q_1 \tag{38}$$

Also in this period the quantity chosen by the firm 1 ($q_1(2)$) is not the optimum choice given the quantity produced by the concurrent, but is closer than the previous-period one to the Nash-equilibrium quantity.

If we continue to repeat this mechanism, we observe that, after some periods, the quantity produced by the firm 1 converges to the Nash-equilibrium quantity. Also the expected profit function converges to a final function (*final expected profit function*).

In fig. 3 (a,b) we can see, on the plane (q_1, π_1) and on the 3D space (q_1, q_2, π_1), the quantities chosen by firm 1 in the first three periods and at the end of the process. In fig. 4 (a,b) we have the corresponding situation of the firm 2. We can also see the expected profit functions ($\pi^e(t)$ is the short form for the expected profit of the firm 1 in fig.3 and for the firm 2 in fig.4) and the section of the real profit function corresponding to the quantity chosen by the firms in the first three periods and at the end of the process ($\pi(t)$ is the short form for $\pi_1|_{q_2=q_2(t)}$ in fig.3 and the short form for $\pi_2|_{q_1=q_1(t)}$ in fig.4).

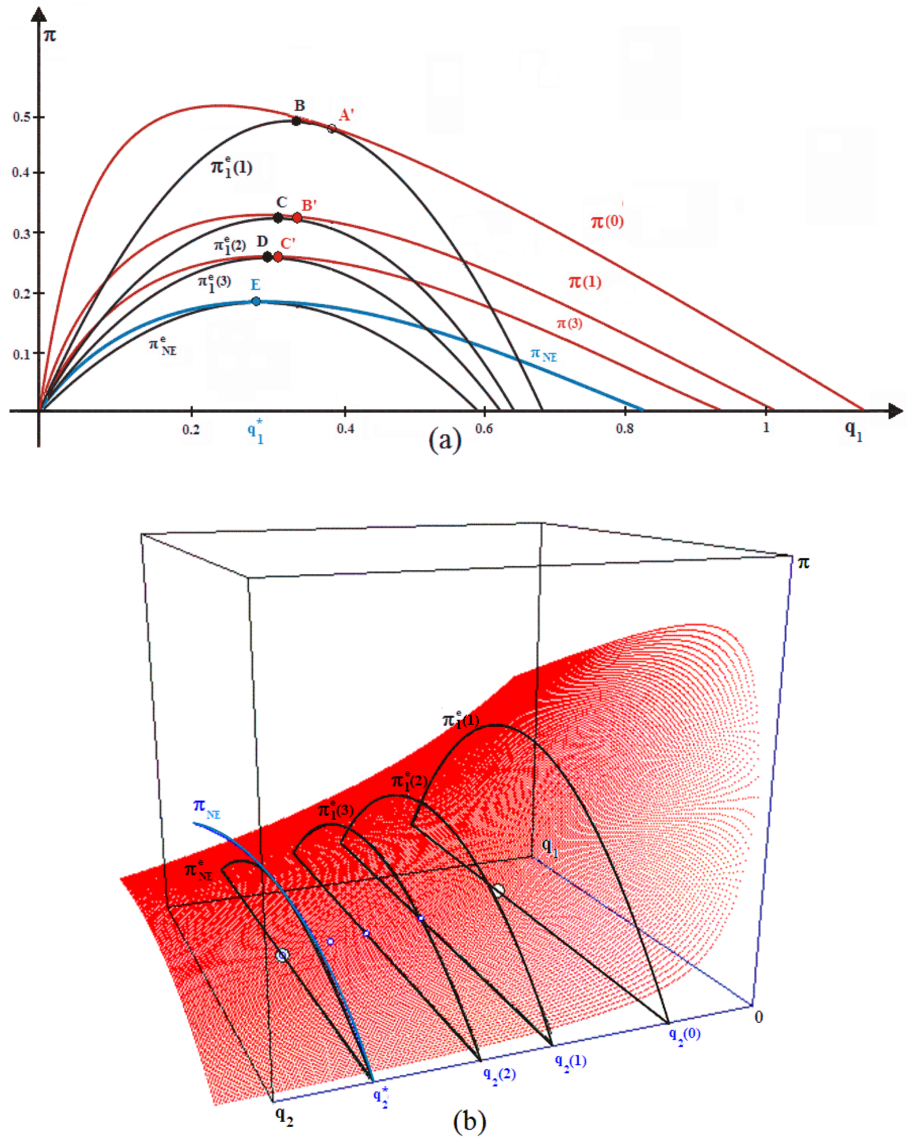

Figure 3: Expected and real profit functions for the firm 1

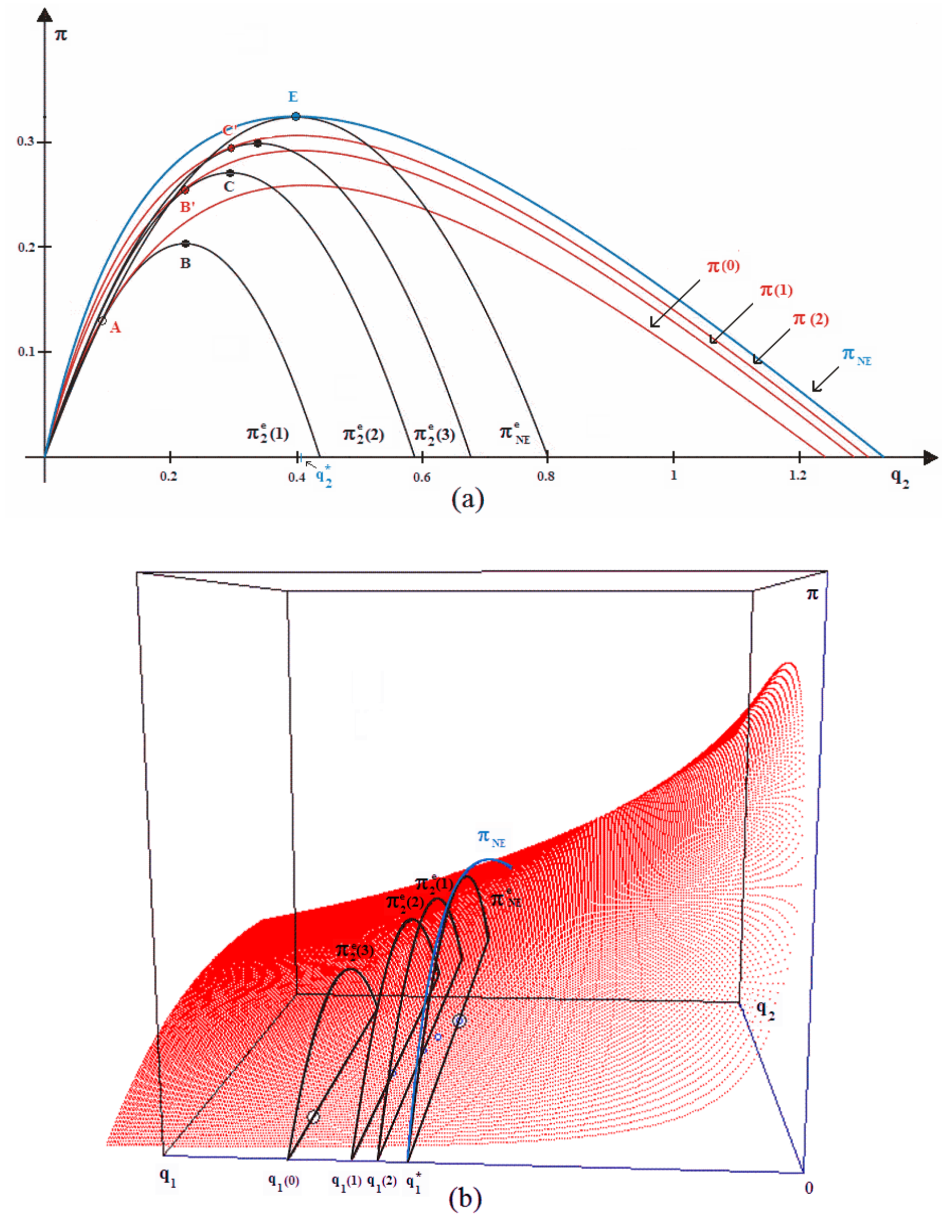

Figure 4: Expected and real profit functions for the firm 2

The point A corresponds to the initial condition and is also the point in which the section of the profit function corresponding to the initial choice of the competitor is tangent to the expected profit function for the next period; in other words firms think that continuing to produce the same quantities they obtain also the same profit (this because every firm expects that the competitor will not change its choice). This is true every time period. The point B is the point in which firm 1 supposes it will be in the period 1, whereas the point B' is the point in which it really is in that period. Points C and C' have the same meaning for the period 2. The curve π_{NE} is the section of the profit function corresponding to the Nash Equilibrium quantity of the competitor and π^e_{NE} is the final expected profit function. These two curves have the same maximum point (which is the Nash Equilibrium quantity) and the same maximum value (the real profit when the two firms produce their Nash Equilibrium quantities). In the next section we try to generalize this result.

6 Comparison between expected profit functions. The case of isoelastic demand function

In this section we compare the expected profits of firms which know the real shape of the demand function with the profits corresponding to firms that operate under local approximation of the demand function. We consider again the case in which the demand function is isoelastic.

In both cases we consider, like in the previous sections, the case of naïve expectations. Before we proceed it could be useful to underscore that firms have to make an expectation both on the shape of the demand function and on the production of the competitor. We analyze the case in which firms adopt naïve expectations concerning the competitor's output but in one case they also need to estimate the demand function (using local approximation), so the comparison is between the classic Cournot-Puu model and our boundedly rational agent model.

6.1 Expected profits in the Cournot-Puu model

At each time period t, firm 1 (for the firm 2 we only need to invert 1 and 2 in what follows) builds an expected profit function for the period $t+1$, given the outputs and the price realized in t:

$$\pi^e_1(t+1) = p^e_1(t+1)q_1(t+1) - C_1(q_1(t+1)) \tag{39}$$

where $C_1(q_1)$ is a generic cost function.

In the Cournot-Puu model the firm knows the shape of the demand function, so using naïve expectations we have:

$$p^e_1(t+1) = \frac{1}{q_1(t+1) + q_2(t)} \tag{40}$$

Substituting the expected price (40) in the expected profit function (39) we obtain:

$$\pi^e_{1,CP}(t+1) = \frac{q_1(t+1)}{q_1(t+1) + q_2(t)} - C_1(q_1(t+1)) \tag{41}$$

6.2 Expected profits with local approximation of the demand function

Let us consider now the case in which firms don't know the shape of the demand function and, at each time period t, use realized outputs and price to build a linear approximation of the demand curve.

In this case, according to (14), the expected price is:

$$p_1^e(t+1) = \frac{1}{q_1(t)+q_2(t)} - \frac{q_1(t+1)-q_1(t)}{[q_1(t)+q_2(t)]^2} \tag{42}$$

Using the expected price (42) in the expected profit function (39), we obtain:

$$\pi_{1,LA}^e(t+1) = \left[\frac{1}{q_1(t)+q_2(t)} - \frac{q_1(t+1)-q_1(t)}{[q_1(t)+q_2(t)]^2}\right] q_1(t+1) - C_1(q_1(t+1)) \tag{43}$$

6.3 The comparison

Now we want to make some considerations on the difference (Δ) between the two expected profit function, that is:

$$\Delta = \pi_{1,CP}^e(t+1) - \pi_{1,LA}^e(t+1) = \left[\frac{1}{q_1(t+1)+q_2(t)} - \frac{1}{q_1(t)+q_2(t)} + \frac{q_1(t+1)-q_1(t)}{[q_1(t)+q_2(t)]^2}\right] q_1(t+1) \tag{44}$$

A first consideration is that the difference does not depend on the shape of the cost function (if the costs of each firm only depends on its own output).

A second (and more obvious) consideration is that $\Delta = 0$ if $q_1(t+1) = 0$. In particular in both the cases, firms know that their revenue is 0 if they don't produce at all.

More interesting is the analysis with positive production. After some algebraic manipulation (and excluding the case $q_1(t+1) = 0$) we can easily prove that:

$$\Delta \geq 0 \Leftrightarrow g(q_1(t+1)) \geq 0 \tag{45}$$

with $g(q_1(t+1)) = (q_1(t+1))^2 - 2q_1(t+1)q_1(t) + (q_1(t))^2$.

The convex parabola $g(q_1(t+1))$ is tangent to the horizontal axis in the point $q_1(t+1) = q_1(t)$ and positive for all the other value of $q_1(t+1)$. So we can conclude that:

Proposition 4 *Given an isoelastic demand function and naïve expectations, the profits expected by a firm that knows the shape of the real demand function, are higher then the ones expected by the same firm if it does not know the demand function and approximate it using local approximation. The expected profits are the same only if* $q_1(t+1) = 0$ *or* $q_1(t+1) = q_1(t)$.

In the next section we prove that this result is the consequence of the convexity of the isoelastic demand function and that the result does not change using another convex demand function.

7 Comparison between expected profit functions. The case of a generic convex demand function

What we have found for the isoelastic demand function is caused by its convexity. We can prove it in a graphic way:

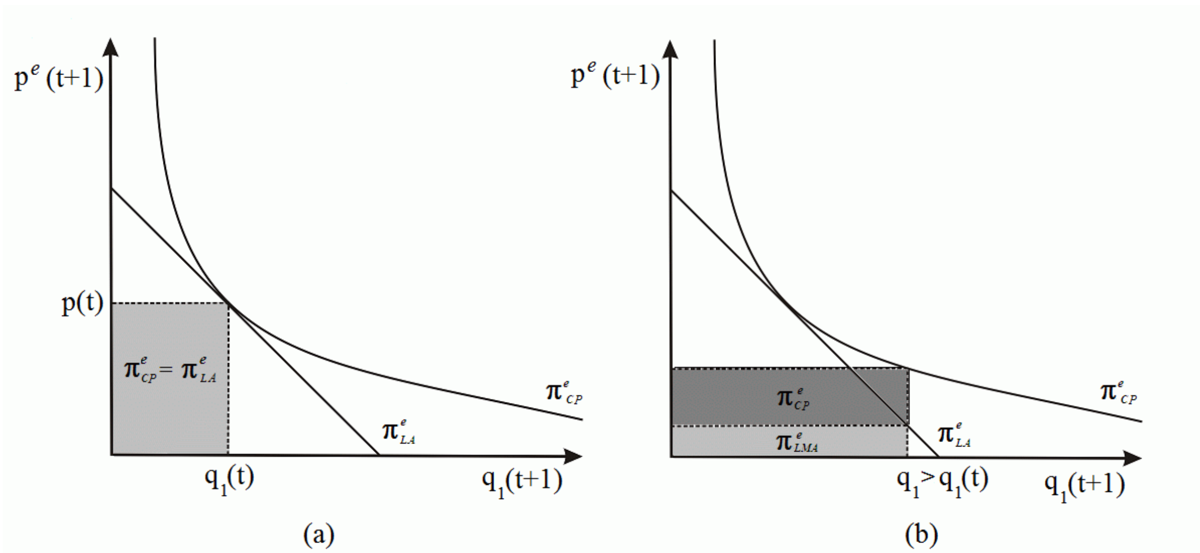

Figure 5: Comparison between expected profits

As we have seen in the previous section the difference between expected profit functions does not depend on the costs, so we can consider a case without them.

In the first picture we can see that if the firm 1 produce at time $t+1$ the same output produced in t, the rectangles, whose area measures the profits in the Cournot-Puu and in the local approximation case are coincident and this happens because the two curves are tangent in that point.

The second picture represents the case in which firm 1 produces in $t+1$ an higher quantity of output with respect to the previous period t. In this case, the convexity of the demand function makes higher the height of the rectangle whose area measure the expected profit in the Cournot-Puu case, whereas the base is the same in both cases. It is easy to see that this is caused both by the convexity of the demand function and the linearity of the expected demand function in the local approximation case. In fact, given that the curves are tangent for $q_1(t+1) = q_1(t)$, for all the other values of $q_1(t+1)$ the convex curve is higher then the linear one (by definition of convexity). So the same happens for all the (positive) values of $q_1(t+1)$ except for $q_1(t+1) = q_1(t)$.

The isoelastic demand function introduced by Puu is a convex function so the result of the previous section is just a consequence of it.

We could easily show that if the demand function is concave then the expected profit in the local approximation case will be always higher then in the Cournot case (except for $q_1(t+1) = q_1(t)$).

8 Conclusions

Our work moves from a well known duopoly model with isoelastic demand function in which the firms are endowed with *naïve* expectations (Puu-Cournot duopoly). In that case for values of the ratio between the marginal costs that are sufficiently high or low, the Cournot-Nash equilibrium loses stability via flip bifurcation, so we can find locally stable period cycles or even a chaotic attractor.

In this study we modify the assumptions concerning the rationality of the firms. In particular, we propose a repeated game in which firms are endowed with a lower degree of rationality and a lower information set than the Puu-Cournot firms. Firms get the correct local estimate of the demand function and then they use such estimate for a linear approximation of the demand function. On the basis of this subjective demand function they solve their profit maximization problem.

We prove that under these alternative assumptions the unique steady state is globally asymptotically stable for any possible configuration of the marginal costs. This result implies that a decreasing in degree of rationality, in this case, does not lead to a destabilization of the model but it has the opposite stabilizing effect.

We also compare the expected profit function of the firms in both cases, proving that for (almost) all the expected values of the output of the concurrent firm, the duopolists, in our model, expect an higher profit than the Puu-Cournot ones. An exception is given by the expected profit on the Nash equilibrium in which they are both equal to the real one and they both maximize the respective expected profit function. We generalize these results to all models with convex demand function.

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Figures