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"Representing Fuzzy Numbers for Fuzzy Calculus"

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Representing fuzzy numbers for fuzzy calculus

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Abstract. In this paper we illustrate the LU representation of fuzzy numbers and present an LU-fuzzy calculator, in order to explain the use of the LU-fuzzy model and to show the advantage of the parametrization. The model can be applied either in the level-cut or in generalized LR frames. The hand-like fuzzy calculator has been developed for the MS-Windows platform and produces the basic fuzzy calculus: the arithmetic operations (scalar multiplication, addition, subtraction, multiplication, division) and the fuzzy extension of many univariate functions (exponential, logarithm, power with numeric or fuzzy exponent, sin, arcsin, cos, arccos, tan, arctan, square root, Gaussian, hyperbolic sinh, cosh, tanh and inverses, erf and erfc error functions, cumulative standard normal distribution).

1 Introduction

The arithmetic operations on fuzzy numbers are usually approached either by the use of the extension principle (in the domain of the membership function, [8]) or by the interval arithmetics (in the domain of the $\alpha - cuts$) as outlined by Dubois and Prade ([1]); the same authors have introduced the well known LR model and the corresponding formulas for the fuzzy operations ([2]); an extensive survey and bibliography is in [3].

In [4], the use of monotonic splines is suggested to approximate fuzzy numbers, using several interpolation forms and a procedure is described to control the error of the approximation. The parametric LU representation allows a large set of possible shapes (types of membership functions) that seems to be much wider than the well-known LR framework (see also [6] and [7]).

The paper is organized as follows: in sections 2 and 3 we describe the LU-fuzzy model and calculus and some example algorithms which implement the LU-fuzzy extension principle for unidimensional elementary functions. Section 5 contains a description of the LU-fuzzy calculator.

1.1 Basic fuzzy calculus

We adopt the so called $a - cut$ setting for the definition of a fuzzy number:

Definition 1. *A continuous fuzzy number (or interval) u is any pair (u^-, u^+) of functions $u^\pm : [0, 1] \longrightarrow \mathbb{R}$ satisfying the following conditions:*

(i) $u^-:\alpha\longrightarrow u^-_\alpha\in\mathbb{R}$ *is a bounded monotonic increasing (non decreasing) continuous function* $\forall\alpha\in[0,1]$;
(ii) $u^+:\alpha\longrightarrow u^+_\alpha\in\mathbb{R}$ *is a bounded monotonic decreasing (non increasing) continuous function* $\forall\alpha\in[0,1]$;
(iii) $u^-_\alpha\le u^+_\alpha$ $\forall\alpha\in[0,1]$.

The notation $u_\alpha=[u^-_\alpha,u^+_\alpha]$ is used explicitly for the $\alpha-$*cuts* of u. We will also refer to u^- and u^+ as the lower and the upper branches on u, respectively. If $u=(u^-,u^+)$ and $v=(v^-,v^+)$ are two given fuzzy numbers, the interval-based arithmetic operations are defined in the usual way, for $\alpha\in[0,1]$:

(Addition) $(u+v)_\alpha=[u^-_\alpha+v^-_\alpha,u^+_\alpha+v^+_\alpha]$.

(Scalar Multiplication) For $k\in\mathbb{R}$, $(ku)_\alpha=[\min\{ku^-_\alpha,ku^+_\alpha\},\max\{ku^-_\alpha,ku^+_\alpha\}]$.

(Subtraction) $(u-v)_\alpha=[u^-_\alpha-v^+_\alpha,u^+_\alpha-v^-_\alpha]$.

(Multiplication) $\begin{cases}(uv)^-_\alpha=\min\{u^-_\alpha v^-_\alpha,u^-_\alpha v^+_\alpha,u^+_\alpha v^-_\alpha,u^+_\alpha v^+_\alpha\}\\(uv)^+_\alpha=\max\{u^-_\alpha v^-_\alpha,u^-_\alpha v^+_\alpha,u^+_\alpha v^-_\alpha,u^+_\alpha v^+_\alpha\}\end{cases}$.

(Division) If $0\notin[v^-_0,v^+_0]$, $\begin{cases}\left(\frac{u}{v}\right)^-_\alpha=\min\left\{\frac{u^-_\alpha}{v^-_\alpha},\frac{u^-_\alpha}{v^+_\alpha},\frac{u^+_\alpha}{v^-_\alpha},\frac{u^+_\alpha}{v^+_\alpha}\right\}\\\left(\frac{u}{v}\right)^+_\alpha=\max\left\{\frac{u^-_\alpha}{v^-_\alpha},\frac{u^-_\alpha}{v^+_\alpha},\frac{u^+_\alpha}{v^-_\alpha},\frac{u^+_\alpha}{v^+_\alpha}\right\}\end{cases}$.

2 LU-fuzzy representation and calculus

The parametric LU representation of a fuzzy number is defined on a decomposition of the interval $[0,1]$, $0=\alpha_0<\alpha_1<.....<\alpha_{i-1}<\alpha_i<....<\alpha_N=1$ for both the lower $u^-(\alpha)$ and the upper $u^+(\alpha)$ branches. In each of the N subintervals $I_i=[\alpha_{i-1},\alpha_i]$, $i=1,2,...,N$, the values of the two functions $u^-(\alpha_{i-1})=u^-_{0,i}$, $u^+(\alpha_{i-1})=u^+_{0,i}$, $u^-(\alpha_i)=u^-_{1,i}$, $u^+(\alpha_i)=u^+_{1,i}$ and their first derivatives $u'^-(\alpha_{i-1})=d^-_{0,i}$, $u'^+(\alpha_{i-1})=d^+_{0,i}$, $u'^-(\alpha_i)=d^-_{1,i}$, $u'^+(\alpha_i)=d^+_{1,i}$ are assumed to be known; we are interested in families of monotonic functions that satisfy the above eight Hermite-type conditions for each subinterval I_i. In general, by transforming each subinterval I_i into the standard $[0,1]$ interval, i.e. $t_\alpha=\frac{\alpha-\alpha_{i-1}}{\alpha_i-\alpha_{i-1}}$, $\alpha\in I_i$, we can determine each piece independently and obtain piecewise continuous LU-fuzzy numbers. Globally continuous or more regular $C^{(1)}$ fuzzy numbers can be obtained directly from the data if the following conditions are met for the values and possibly for the slopes:

$$u^-_{1,i}=u^-_{0,i+1},\ \ u^+_{1,i}=u^+_{0,i+1},\ \ d^-_{1,i}=d^-_{0,i+1},\ \ d^+_{1,i}=d^+_{0,i+1},\ \ \text{for } i=1,2,...,N-1.$$

Let $p_i(t_\alpha)$ be a model function for u_α on a generic subinterval I_i; then, for $t_\alpha\in[0,1]$ we have

$$\begin{aligned}p_i(t_\alpha)&=u(\alpha_{i-1}+t_\alpha(\alpha_i-\alpha_{i-1}))\\p'_i(t_\alpha)&=u'(\alpha_{i-1}+t_\alpha(\alpha_i-\alpha_{i-1}))(\alpha_i-\alpha_{i-1}).\end{aligned}\tag{1}$$

Proposed $p(t)$ functions are the (2,2)-rational monotonic spline

$$
\begin{aligned}
p(t) &= \begin{cases} \frac{P(t)}{Q(t)} & \text{if } u_1 \neq u_0 \\ u_0 & \text{if } u_1 = u_0 \end{cases}, \quad \text{where} \\
P(t) &= (u_1 - u_0)\, u_1 t^2 + (u_0 d_1 + u_1 d_0)\, t(1-t) + (u_1 - u_0)\, u_0\, (1-t)^2 \\
Q(t) &= (u_1 - u_0)\, t^2 + (d_1 + d_0)\, t(1-t) + (u_1 - u_0)\, (1-t)^2 \,;
\end{aligned}
$$

the (3,2)-rational monotonic spline

$$
\begin{aligned}
p(t) &= \frac{P(t)}{Q(t)} \text{ with} \\
P(t) &= u_0\, (1-t)^3 + (w u_0 + d_0)\, t\, (1-t)^2 + (w u_1 - d_1)\, t^2\, (1-t) + u_1 t^3 \\
Q(t) &= 1 + t\, (1-t)\, (w-3) \\
w &= \frac{d_0 + d_1}{u_1 - u_0} \text{ to have monotonicity;}
\end{aligned}
$$

and the monotonic mixed cubic-exponential spline

$$
\begin{aligned}
p(t) &= u_0 + (u_1 - u_0 - \frac{d_0 + d_1}{a}) t^2 (3 - 2t) + \frac{d_0}{a} - \frac{d_0}{a}(1-t)^a + \frac{d_1}{a} t^a \\
a &= 1 + \frac{d_0 + d_1}{u_1 - u_0} \text{ to have monotonicity.}
\end{aligned}
$$

The models include linear (i.e. triangular fuzzy numbers), monotonic quadratic and monotonic cubic polynomials as special cases.

Using one of the previous forms to represent the lower and the upper branches of the fuzzy number $u = (u^-, u^+)$ we can write the general form of the representation (the symbol δ is used to denote the slopes or first derivatives).

$$
\begin{aligned}
u &= (u^-_{0,i}, \delta u^-_{0,i}, u^-_{1,i}, \delta u^-_{1,i}; u^+_{0,i}, \delta u^+_{0,i}, u^+_{1,i}, \delta u^+_{1,i})_{i=1,\dots,N} \\
&\Updownarrow \\
u_\alpha &= [p_i(t_\alpha; u^-_{0,i}, \widetilde{\delta u}^-_{0,i}, u^-_{1,i}, \widetilde{\delta u}^-_{1,i}), p_i(t_\alpha; u^+_{0,i}, \widetilde{\delta u}^+_{0,i}, u^+_{1,i}, \widetilde{\delta u}^+_{1,i})]_{i=1,2,\dots,N}
\end{aligned} \tag{2}
$$

with $\widetilde{\delta u}_{k,i} = \delta u_{k,i}(\alpha_i - \alpha_{i-1})$, $k = 0, 1$. For $N \geq 1$ we have a total of $8N$ parameters $u^-_{0,1} \leq u^-_{1,1} \leq u^-_{0,2} \leq u^-_{1,2} \leq \dots \leq u^-_{0,N} \leq u^-_{1,N}$, $\delta u^-_{k,i} \geq 0$ defining the increasing lower branch u^-_α and $u^+_{0,1} \geq u^+_{1,1} \geq u^+_{0,2} \geq u^+_{1,2} \geq \dots \geq u^+_{0,N} \geq u^+_{1,N}$, $\delta u^+_{k,i} \leq 0$ defining the decreasing upper branch u^+_α (obviously, also $u^-_{1,N} \leq u^+_{1,N}$ is required).

A simplification of (2) can be obtained by requiring differentiable branches: $u^-_{1,i} = u^-_{0,i+1}$, $u^+_{1,i} = u^+_{0,i+1}$ and $\delta u^-_{k,i} = \delta u^-_{k,i+1}$, $\delta u^+_{k,i} = \delta u^+_{k,i+1}$. The number of parameters is reduced to $4N + 4$ and we can write

$$
\begin{aligned}
& u = (u^-_i, \delta u^-_i, u^+_i, \delta u^+_i)_{i=0,1,\dots,N} \text{ with the data} \\
& u^-_0 \leq u^-_1 \leq \dots \leq u^-_N \leq u^+_N \leq u^+_{N-1} \leq \dots \leq u^+_0 \text{ and the slopes} \\
& \delta u^-_i \geq 0, \delta u^+_i \leq 0.
\end{aligned} \tag{3}
$$

3 Arithmetic operations

Given $u = (u_i^-, \delta u_i^-, u_i^+, \delta u_i^+)_{i=0,1,\ldots,N}$ and $v = (v_i^-, \delta v_i^-, v_i^+, \delta v_i^+)_{i=0,1,\ldots,N}$, the arithmetic operators associated to the LU representation can be obtained easily.

$$\left\{ u + v = (u_i^- + v_i^-, \delta u_i^- + \delta v_i^-, u_i^+ + v_i^+, \delta u_i^+ + \delta v_i^+)_{i=0,1,\ldots,N} \right.$$

$$\begin{cases} ku = (ku_i^-, k\delta u_i^-, ku_i^+, k\delta u_i^+)_{i=0,1,\ldots,N}, & \text{if } k \geq 0 \\ ku = (ku_i^+, k\delta u_i^+, ku_i^-, k\delta u_i^-)_{i=0,1,\ldots,N}, & \text{if } k < 0 \end{cases}$$

$$\left\{ u - v = u + (-v) \right.$$

$$\begin{cases} w = uv = \left(w_i^-, \delta w_i^-, w_i^+, \delta w_i^+\right)_{i=0,1,\ldots,N} \text{ with} \\ w_i^- = \min\{u_i^- v_i^-, u_i^- v_i^+, u_i^+ v_i^-, u_i^+ v_i^+\} \\ w_i^+ = \max\{u_i^- v_i^-, u_i^- v_i^+, u_i^+ v_i^-, u_i^+ v_i^+\} \\ w_i^- = u_i^{p_i^-} v_i^{q_i^-} \text{ and } w_i^+ = u_i^{p_i^+} v_i^{q_i^+} \\ \delta w_i^- = \delta u_i^{p_i^-} v_i^{q_i^-} + u_i^{p_i^-} \delta v_i^{q_i^-}, \ \delta w_i^+ = \delta u_i^{p_i^+} v_i^{q_i^+} + u_i^{p_i^+} \delta w_i^{q_i^+} \end{cases}$$

$$\begin{cases} z = u/v = \left(z_i^-, \delta z_i^-, z_i^+, \delta z_i^+\right)_{i=0,1,\ldots,N} \text{ with} \\ (u/v)_i^- = \min\{u_i^-/v_i^-, u_i^-/v_i^+, u_i^+/v_i^-, u_i^+/v_i^+\} \\ (u/v)_i^+ = \max\{u_i^-/v_i^-, u_i^-/v_i^+, u_i^+/v_i^-, u_i^+/v_i^+\} \\ z_i^- = u_i^{r_i^-}/v_i^{s_i^-} \text{ and } z_i^+ = u_i^{r_i^+}/v_i^{s_i^+} \\ \delta z_i^- = (\delta u_i^{r_i^-} v_i^{s_i^-} - u_i^{r_i^-} \delta v_i^{s_i^-})/(v_i^{s_i^-})^2, \ \delta z_i^+ = (\delta u_i^{r_i^+} v_i^{s_i^+} - u_i^{r_i^+} \delta v_i^{s_i^+})/(v_i^{s_i^+})^2 \end{cases}$$

where, for the *multiplication*, (p_i^-, q_i^-) is the pair of superscripts $+$ and $-$ giving the minimum $(uv)_i^-$ and similarly (p_i^+, q_i^+) is the pair of $+$ and $-$ giving the maximum $(uv)_i^+$; analogous symbols can be deduced for the *division,* (r_i^-, s_i^-) is the pair of $+$ and $-$ giving the minimum in $(u/v)_i^-$ and (r_i^+, s_i^+) is the pair of $+$ and $-$ giving the maximum in $(u/v)_i^+$.

As pointed out by the results of the experimentation reported in [4], the operations above are exact at the nodes α_i of the representation and have very small global errors on $[0, 1]$. Further, it is easy to control the error by introducing additional nodes into the representation or by using a sufficiently high number of nodes with $\max\{\alpha_i - \alpha_{i-1}\}$ sufficiently small. To control the error of the approximation, we can proceed by increasing the number $N + 1$ of points; a possible strategy is to double the number of points by using $N = 2^K$ and by moving automatically to $N = 2^{K+1}$ if a better precision is necessary.

The results in [4] of the parametric operators have shown that both the rational and the mixed models perform well with small $N \leq 4$, with a percentage average error for a single multiplication and division of the order of 0.1%.

3.1 Fuzzy extension of univariate functions

The fuzzy extension of a single (real) variable (differentiable) function $f : \mathbb{R} \to \mathbb{R}$ to a fuzzy argument $u_\alpha = [u_\alpha^-, u_\alpha^+]$ has $\alpha - cuts$

$$f(u)_\alpha = [\min\{f(x) \mid x \in u_\alpha\}, \max\{f(x) \mid x \in u_\alpha\}]. \tag{4}$$

If f is monotonic increasing we obtain $f(u)_\alpha = [f(u_\alpha^-), f(u_\alpha^+)]$ while, if f is monotonic decreasing, $f(u)_\alpha = [f(u_\alpha^+), f(u_\alpha^-)]$.

Let X be the LU-fuzzy number $X = \left(x_i^-, \delta x_i^-, x_i^+, \delta x_i^+\right)_{i=0,1,\dots,N}$; then its image $Y = f(X) = \left(y_i^-, \delta y_i^-, y_i^+, \delta y_i^+\right)_{i=0,1,\dots N}$ is calculated as follows: let $\widehat{x}_i^- \in [x_i^-, x_i^+]$ and $\widehat{x}_i^+ \in [x_i^-, x_i^+]$ be the points where $\min\left\{f(x) \mid x \in [x_i^-, x_i^+]\right\}$ and $\max\left\{f(x) \mid x \in [x_i^-, x_i^+]\right\}$ are attained; possibly, $\widehat{x}_i^-$, $\widehat{x}_i^+$ are one of the extremes x_i^-, x_i^+ of the interval or may be internal points (where the derivative of f is zero). We then have

$$
\begin{aligned}
y_i^- &= f(\widehat{x}_i^-) \\
y_i^+ &= f(\widehat{x}_i^+) \\
\delta y_i^- &= \begin{cases} f'(\widehat{x}_i^-)\delta x_i^- & \text{if } \widehat{x}_i^- = x_i^- \text{ is the left extreme point of the internal} \\ f'(\widehat{x}_i^-)\delta x_i^+ & \text{if } \widehat{x}_i^- = x_i^+ \text{ is the right extreme point of the internal} \\ 0 & \text{if } \widehat{x}_i^- \in]x_i^-, x_i^+[\text{ is an internal point} \end{cases} \\
\delta y_i^+ &= \begin{cases} f'(\widehat{x}_i^+)\delta x_i^- & \text{if } \widehat{x}_i^+ = x_i^- \text{ is the left extreme point of the internal} \\ f'(\widehat{x}_i^+)\delta x_i^+ & \text{if } \widehat{x}_i^+ = x_i^+ \text{ is the right extreme point of the internal} \\ 0 & \text{if } \widehat{x}_i^+ \in]x_i^-, x_i^+[\text{ is an internal point} \end{cases}
\end{aligned}
$$

Example 1: fuzzy extension of hyperbolic cosinusoidal function

Let

$$Y = \cosh(X) = \frac{e^X + e^{-X}}{2}$$

For each $i = 0, 1, ..., N$:

$$
\begin{aligned}
&\text{if } X_i^+ \le 0 \text{ then } \begin{cases} Y_i^- = \cosh\left(X_i^+\right) \\ Y_i^+ = \cosh\left(X_i^-\right) \\ \delta Y_i^- = \delta X_i^+ \sinh\left(X_i^+\right) \\ \delta Y_i^+ = \delta X_i^- \sinh\left(X_i^-\right) \end{cases} \\
&\text{else if } X_i^+ \le 0 \text{ then } \begin{cases} Y_i^- = \cosh\left(X_i^-\right) \\ Y_i^+ = \cosh\left(X_i^+\right) \\ \delta Y_i^- = \delta X_i^- \sinh\left(X_i^-\right) \\ \delta Y_i^+ = \delta X_i^+ \sinh\left(X_i^+\right) \end{cases} \\
&\text{else } \begin{cases} Y_i^- = 1, \quad \delta Y_i^- = 0 \\ \text{if } abs(X_i^-) \ge abs(X_i^+) \begin{array}{l} \text{then} \begin{cases} Y_i^+ = \cosh\left(X_i^-\right) \\ \delta Y_i^+ = \delta X_i^- \sinh\left(X_i^-\right) \end{cases} \\ \text{else} \begin{cases} Y_i^+ = \cosh\left(X_i^+\right) \\ \delta Y_i^+ = \delta X_i^+ \sinh\left(X_i^+\right) \end{cases} \end{array} \end{cases}
\end{aligned}
$$

Example 2: fuzzy extension of erf and $erfc$ error functions

Let

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-t^2\right) dt = \qquad \text{(increasing)}$$

$$= 1 - \frac{2}{\sqrt{\pi}} \int_x^{+\infty} \exp\left(-t^2\right) dt = 1 - \operatorname{erf} c\,(x) \ \text{ with}$$

$$\operatorname{erf} c\,(x) = \frac{2}{\pi} \int_x^{+\infty} \exp\left(-t^2\right) dt \qquad \text{(decreasing).}$$

We use the following approximation, having a fractional error less than 1.2×10^{-7} :

$$z = abs\,(x)$$

$$t = \frac{1}{1 + \frac{1}{2} z}$$

$$\operatorname{erf} c = \begin{cases} t \exp\left(-z^2 + p\,(t)\right) & if\ x \geq 0 \\ 2 - t \exp\left(-z^2 + p\,(t)\right) & if\ x < 0 \end{cases} \quad \text{with}$$

$$p\,(t) = a_0 + t\,(a_1 + t\,(a_2 + t\,(a_3 + t\,(a_4 + t\,(a_5 + t\,(a_6 + t\,(a_7 + t\,(a_8 + t a_9))))))))$$

$$\begin{array}{ll} a_0 = -1.26551223 & a_5 = 0.27886807 \\ a_1 = 1.00002368 & a_6 = -1.13520398 \\ a_2 = 0.37409196 & a_7 = 1.48851587 \\ a_3 = 0.09678418 & a_8 = -0.82215223 \\ a_4 = -0.18628806 & a_9 = 0.17087277 \end{array}$$

Let $Y = \operatorname{erf}(X)$. For each $i = 0, 1, ..., N$

$$\begin{cases} Y_i^- = \operatorname{erf}\left(X_i^-\right) \\ Y_i^+ = \operatorname{erf}\left(X_i^+\right) \\ \delta Y_i^- = \delta X_i^- \frac{2}{\sqrt{\pi}} \exp\left(-X_i^-\right)^2 \\ \delta Y_i^+ = \delta X_i^+ \frac{2}{\sqrt{\pi}} \exp\left(-X_i^+\right)^2 \end{cases}$$

Let now $Y = \operatorname{erf} c\,(X)$. For each $i = 0, 1, ..., N$

$$\begin{cases} Y_i^- = \operatorname{erf} c\left(X_i^+\right) \\ Y_i^+ = \operatorname{erf} c\left(X_i^-\right) \\ \delta Y_i^- = -\delta X_i^+ \frac{2}{\sqrt{\pi}} \exp\left(-X_i^+\right)^2 \\ \delta Y_i^+ = -\delta X_i^- \frac{2}{\sqrt{\pi}} \exp\left(-X_i^-\right)^2 \end{cases}$$

The cumulative normal function $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt$, $x \in \mathbb{R}$, can be calculated by

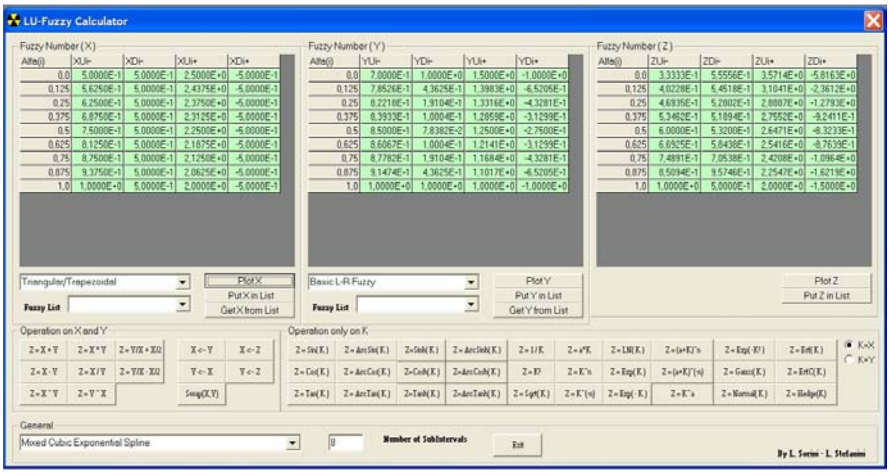
$$\Phi(x) = \begin{cases} \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right) & \text{if } x \geq 0 \\ \\ \frac{1}{2}\left(1 - \operatorname{erf}\left(-\frac{x}{\sqrt{2}}\right)\right) & \text{if } x < 0 \end{cases}$$

Let $Y = \Phi(X)$. For each $i = 0, 1, ..., N$

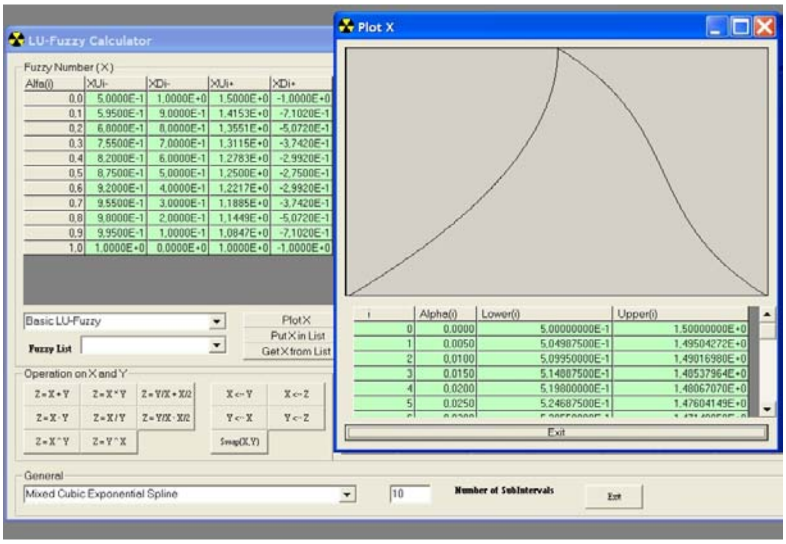
$$\begin{cases} Y_i^- = \Phi\left(X_i^-\right) \\ Y_i^+ = \Phi\left(X_i^+\right) \\ \delta Y_i^- = \delta X_i^- \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-X_i^{-2}}{2}\right) \\ \delta Y_i^+ = \delta x_i^+ \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-X_i^{+2}}{2}\right) \end{cases}$$

4 Implementation of the LU-fuzzy calculator

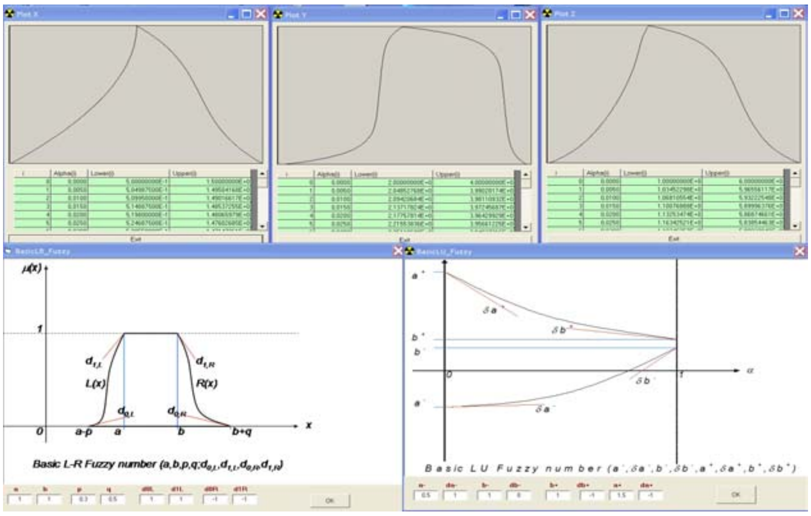
A hand-like fuzzy calculator has been implemented by a Windows-based frame.



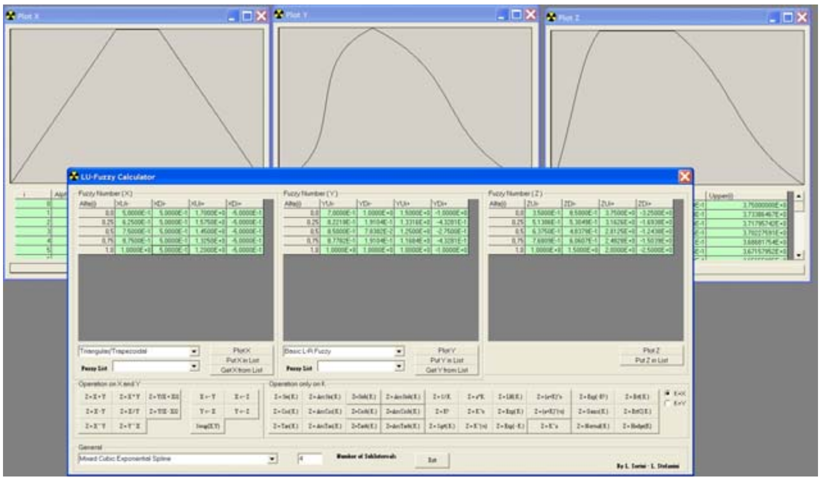
It works by first defining input fuzzy numbers X and Y using the LU-fuzzy representation and produces Z as result of operations. Three boxes are designed to contain the LU-fuzzy representation (grid) of the fuzzy numbers X, Y and Z.



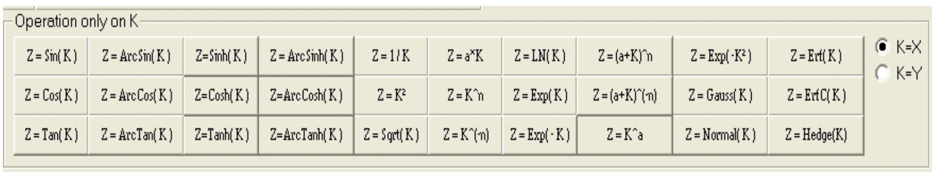
For each element $u \in \{X, Y, Z\}$, the grid box contains the LU-values α_i, u_i^-, δu_i^-, u_i^+ and δu_i^+ respectively. To start the calculations, we have implemented a set of predefined types, including triangular, trapezoidal, general parametrized LU and LR fuzzy numbers.



For any given type, it is possible to define the number N of subintervals ($N+1$ points) in the uniform $\alpha-$decomposition: all the calculations are performed exactly at the nodes of the decomposition and the monotonic splines are then used to interpolate at other values of $\alpha \in [0, 1]$. It is possible to plot the membership functions of the inputs, the intermediate or final results. The Plot button opens a popup window with the graph of the membership function of the corresponding fuzzy number. To obtain the graphs or other representations, one of the models (rational or mixed monotonic splines) can be selected.

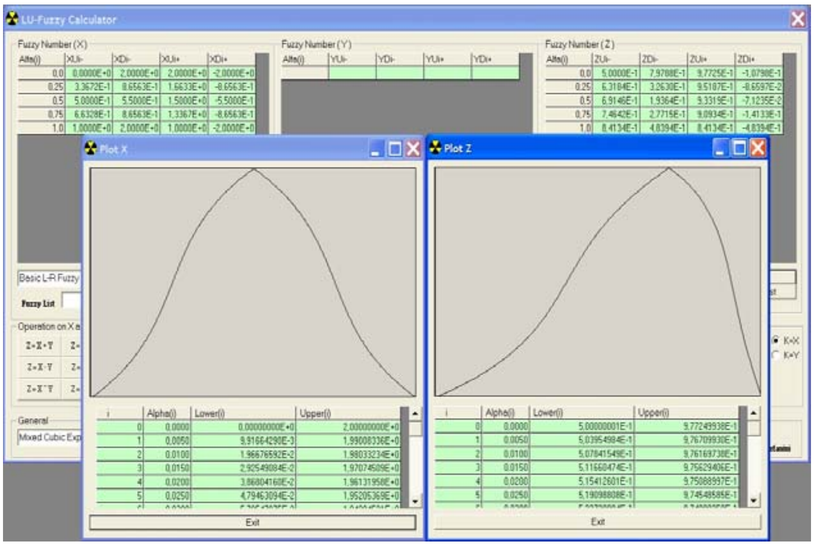


The standard arithmetic operations $Z = X + Y$, $Z = X - Y$, $Z = X * Y$, $Z = X/Y$ and the fuzzy extension of many elementary unidimensional functions are included. The actual implemented functions are $Z = X^Y$, $Z = Y^X$ and, choosing $K = X$ or $K = Y$, $Z = \sin(K)$, $Z = \arcsin(K)$, $Z = \cos(K)$, $Z = \arccos(K)$, $Z = \tan(K)$, $Z = \arctan(K)$, $Z = \sinh(K)$, $Z = \sinh^{-1}(K)$, $Z = \cosh(K)$, $Z = \cosh^{-1}(K)$, $Z = \tanh(K)$, $Z = \tanh^{-1}(K)$, $Z = 1/K$, $Z = aK$ $(a \in \mathbb{R})$, $Z = K^2$, $Z = K^{\pm n}$ $(n \in \mathbb{N})$, $Z = \sqrt{K}$, $Z = \ln(K)$, $Z = \exp(K)$, $Z = \exp(-K)$, $Z = (a + K)^{\pm n}$, $Z = K^a$, $Z = \exp(-K^2)$, $Z = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}K^2)$, $Z = \operatorname{erf}(K)$, $Z = \operatorname{erfc}(K)$, $Z = Normal(K) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{K} \exp(-\frac{1}{2}t^2)dt$. Finally, some Hedge linguistic fuzzy operators are implemented (very, more or less, ...). The calculations are performed by clicking the button of the corresponding operation. The left group of buttons involves the binary operations. The second group of operators require the assigment of either X or Y to the temporary K and operate on K itself putting the result into Z.



It is possible to save a given (X, Y or Z) temporary result into a stored list (Put in List button), by assigning a name to it; a saved fuzzy number can be reloaded either in X or Y for further use (Get from List button). The data are saved into a formatted file having the same user-defined name.

We illustrate an example $Z = Normal(X)$. First select a type of fuzzy number (trapezoidal, LU or LR) and set the number N of subintervals in the $\alpha-$decomposition (the higher N the higher the precision in the calculations); the maximal value of N is 100; typical values are 2, 4, 8, 10. If the selection is loaded into the X-area, the corrisponding grid appears. To see the membership function of X, click the corresponding Plot button and a popup window appears. To apply the fuzzy extension to X, first select the assigment $K = X$ and then click the $Z = Normal(K)$ button.



A detailed description of the calculator is in [5].

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